Towards Kleene Algebra with Recursion

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Abstract

We extend Kozen's theory *KA* of Kleene Algebra to axiomatize parts of the equational theory of context-free languages, using a least fixed-point operator μ instead of Kleene's iteration operator *.

Although the equational theory of context-free languages is not recursively axiomatizable, there are natural axioms for subtheories $KAF \subseteq KAR \subseteq KAG$: respectively, these make μ a least fixed point operator, connect it with recursion, and express S. Greibach's method to replace left- by right-recursion and vice versa. Over KAF, there are different candidates to define * in terms of μ, such as tail-recursion and reflexive transitive closure. In KAR, these candidates collapse, whence KAR uniquely defines * and extends Kozen's theory KA.

We show that a model $\mathcal{M} = (M, +, 0, \cdot, 1, \mu)$ of KAF is a model of KAG, whenever the partial order $\leq$ on M induced by $+$ is complete, and $+$ and $\cdot$ are Scott-continuous with respect to $\leq$. The family of all context-free languages over an alphabet of size n is the free structure for the class of submodels of continuous models of KAF in n generators.

1 Introduction

Regular algebra is the equational theory of the algebra of regular languages, initiated by Kleene[6]. Redko[13] showed that this theory is not axiomatizable by finitely many equations between regular expressions. Recently, two finite axiomatizations by other means have been given. Pratt[12]'s theory of *Action Logic*, ACT, enriches Kleene's set $\{+, 0, \cdot, 1, ^*\}$ of regular operations by left and right *residuals* $\leftarrow$ and $\rightarrow$, and is axiomatized by finitely many equations. Iteration * is characterized in ACT by its monotonicity properties and the equation $(x \rightarrow x)^* = x \rightarrow x$ of 'pure induction'. Kozen[8]'s theory of *Kleene Algebra*, *KA*, sticks to Kleene's regular operations, but characterizes * by universal Horn-axioms. Both ACT and *KA* are complete for regular algebra.

One of the motivations behind these axiomatizations was to provide an alternative to the largely combinatorial constructions of current treatments of the theory of regular languages. We believe that reasoning about context-free languages could also profit from algebraic and logical means based on axiomatic theories. In particular, it seems that in studying context-free languages, the algebraic tool of formal power series could sometimes be replaced by simpler logical methods exploiting properties of a least-fixed point operator. The aim of this paper is to stimulate research in this direction by giving some basic considerations and problems.

Recall that *regular expressions* are inductively defined by

$$r \quad := \quad 0 \quad | \quad 1 \quad | \quad x \quad | \quad a \quad | \quad (r+r) \quad | \quad (r\cdot r) \quad | \quad r^* \ ,$$

where a ranges over a finite list or *alphabet* Σ of constants, and x over an infinite list of variables. With a least-fixed-point operator μ, we define *μ-regular expressions* by

$$r \quad := \quad 0 \quad | \quad 1 \quad | \quad x \quad | \quad a \quad | \quad (r+r) \quad | \quad (r\cdot r) \quad | \quad r^* \quad | \quad \mu x\, r.$$

There are two *standard interpretations* of regular expressions:

In the *language interpretation*, $\mathcal{L}_\Sigma$, variables range over the universe of all subsets (or *formal languages*) of the set Σ^* of finite sequences (or *words*) of elements of Σ, 0 denotes the empty set, 1 the singleton set containing the empty word ϵ only, the constant a denotes $\{a\}$, $+$ is set union, $\cdot$ is element-wise concatenation, and * is the union of all finite concatenations of a language with itself. By $\mathcal{REG}_\Sigma$ we understand the subclass of all regular languages over Σ, which are those elements of $\mathcal{L}_\Sigma$ that are the value of a closed regular expression, i.e. one without free variables.

In the *relation interpretation*, $\mathcal{R}_K$, variables range over the universe of all binary relations on a set K, with the empty relation as 0, the identity on K as 1, union and composition as $+$ and $\cdot$, and reflexive transitive closure as *. As an interpretation for the constants a we can take any relations $R_a \subseteq K \times K$; however, as there is no canonical choice for these, it seems natural here to consider *pure* expressions only, i.e. those containing no constants except 0 and 1.

Since all operations are monotone with respect to set and relation inclusion, respectively, we can extend these standard interpretations to μ-regular expressions, letting μ pick the least fixed point of monotone functions. The fundamental theorem of recursion theory on the natural numbers, saying that *(i)* a partial function f is definable by a system of Gödel-Herbrand-Kleene-equations iff *(ii)* f is μ-recursively definable iff *(iii)* f is computable by a Turing-machine, has the following analogue concerning the definability of formal languages[1]:

Theorem 1.1 *For every language $A \subseteq \Sigma^*$, the following conditions are equivalent:*

(i) A is definable by a system of regular equations,

(ii) A is definable by a μ-regular expression,

(iii) A is accepted by a pushdown-store automaton.

[1]In characterizations of context-free languages in the literature, condition (ii) often is missing. This is true at least for Hopcroft/Ullman[5], Lewis/Papadimitriou[9], and Harrison[4]. Salomaa[15] has (ii), but in a somewhat less persipcious notation. Essentially, the pushdown store is just the procedure return stack by which the finite automata for the r_i organize the 'calling' of each other, i.e. their transitions labelled with variables. This offers a nice and simple way to introduce pushdown-store automata and to explain why an implementation of recursion μ needs a stack.

Here, *definable by a system of regular equations* means to be a component of the least solution of a system

$$\begin{array}{ccc} x_1 & = & r_1(x_1, \ldots, x_m) \\ \vdots & \vdots & \vdots \\ x_m & = & r_m(x_1, \ldots, x_m), \end{array}$$

where each r_i is a regular expression whose free variables are among the pairwise distinct recursion variables $x_1, \ldots, x_m$. Context-free grammars can be seen as systems $\{x_i = r_i\}_{1 \leq i \leq m}$ of regular equations where the r_i are *-free and in disjunctive normal form. One can use more restrictive normal forms, such as Greibach's[3], or more general expressions, such as μ-regular ones, without changing the class of languages that are definable by systems of equations.

In the present paper, we will use μ-regular expressions as a naming system for context-free languages[2] to axiomatize pieces of their equational theory. This will be done in three steps:

First, we add to the algebraic properties of $+, 0, \cdot, 1$ the assumption that μ picks the least pre-fixed point of definable functions $\lambda x.r$, for each μ-regular expression $r(x, y_1, \ldots, y_n)$:

$$\forall y_1 \ldots \forall y_n \ (r[\mu x.r/x] \leq \mu x.r \ \wedge \ \forall x(r \leq x \to \mu x.r \leq x)). \tag{1}$$

To express minimality, we need universal Horn-axioms, with conditional (in)equations in the quantifier free part. The resulting theory will be called *Kleene algebra with least fixed points*, or *KAF* for short. (Actually, this is a misnomer, if we drop * from the language).

In a second step, we extend *KAF* by equational axioms to obtain a theory of *Kleene algebra with recursion*, *KAR*, that can model iteration * by recursion μ, a suitably restricted form of the least fixed point operator. In *KAR*, we relate μ with the algebraic operations $+$ and $\cdot$ by

$$\forall a, b. \ (\mu x(b + ax) = \mu x(1 + xa) \cdot b \quad \wedge \quad \mu x(b + xa) = b \cdot \mu x(1 + ax)). \tag{2}$$

(In the following, we often write a, b for free variables, while a_i will be a constant of Σ that may be added to the pure language, without adding any relations between these.) Using these equations and taking $\mu x(1 + ax)$ as a^*, the least fixed point properties for $\mu x(b + ax)$ and $\mu x(b + xa)$ in *KAF* are just the properties of * taken as axioms in Kozen's[8] theory *KA* of Kleene algebra. Hence, *KAR* is indeed a theory of *Kleene* algebras, with iteration generalized to recursion. While *KAR* proves all equations between regular expressions that are valid in the language interpretation, it cannot prove all valid equations between μ-regular expressions, nor can it be completed to do so.

In spite of this limitation, in a third step we look for larger fragments of the equational theory of context-free languages that have natural axioms. We note that

[2] Niwinski[10] investigates the hierarchy of ω-languages obtained by nested least and largest fixed point operators.

S.Greibach's way to eliminate left recursion in context-free grammars relies on an equivalence between grammars that can compactly be expressed as an equation schema between μ-regular expressions r, s, possibly containing x:

$$\mu x(s + rx) = \mu x(\mu y(1 + yr) \cdot s) \quad \wedge \quad \mu x(s + xr) = \mu x(s \cdot \mu y(1 + ry)). \qquad (3)$$

Adding this schema to KAF gives a theory KAG, which extends KAR. It remains to be seen to what extent common reasoning about the equivalence between context-free grammars can be carried out within KAG.

Equations (2) can also be read as claiming continuity properties about $+$ and $\cdot$. We consider continuous models of KAF and relate these to Conway's[2] notion of *Standard Kleene Algebra*. It will be shown that all continuous models of KAF, in particular the standard interpretations, satisfy the identities (3). Moreover, an equation between μ-regular expressions that is valid in the interpretation by context-free languages holds universally in the continuous models of KAF.

2 Kleene algebras with least fixed points

Kleene[6] introduced regular expressions and showed that their equivalence (i.e. equality in the language interpretation) is decidable via finite automata. Several attempts have been made to give a complete axiomatization of this equivalence, notably by Salomaa[14], Conway[2], Pratt[12], and Kozen[8]. The main obstacle was Redko's[13] result that there can be no finite axiomatization by means of equations between regular expressions. Recently, Kozen[8] presented the following axiomatization using Horn-formulas.

Definition 2.1 (D. Kozen) The theory of *Kleene algebras*, *KA*, stated in the language $\{+, 0, \cdot, 1, ^*\}$, consists of (1) the theory of idempotent semirings, which says that

- $+$ is associative, commutative, idempotent, and has 0 as neutral element,
- $\cdot$ is associative and has 1 as neutral element from both sides,
- 0 is an annihilator for $\cdot$ from both sides, i.e. $\forall x(0 \cdot x = 0 = x \cdot 0)$,
- $\cdot$ distributes over $+$ from both sides,

and (2) the following assumptions about *, universally quantified over a, b:

$$1 + aa^* \leq a^*, \quad \text{and} \quad \forall x(b + ax \leq x \;\rightarrow\; a^*b \leq x), \qquad (4)$$
$$1 + a^*a \leq a^*, \quad \text{and} \quad \forall x(b + xa \leq x \;\rightarrow\; ba^* \leq x). \qquad (5)$$

We use $a \leq b$ as shorthand for $b = (a + b)$. Since $+$ is idempotent, associative, and commutative, the relation $\leq$ is reflexive, transitive, and anti-symmetric, i.e. a partial ordering.

Several other notions of Kleene algebra have been studied in the literature; c.f. Conway[2], Büchi[1], and Pratt[11, 12]. Kozen[8], Theorem 5.5, shows that *KA* is equationally complete with respect to the language interpretation: for all closed regular expressions r and s over Σ,

$$KA \vdash r = s \qquad \text{if and only if} \qquad \mathcal{REG}_\Sigma \models r = s,$$

where $\vdash$ is provability in first-order logic.

A model of the equational theory of $\mathcal{REG}_\Sigma$ is called a *regular algebra*. By Theorem 1.1, the closed μ-regular expressions define exactly the context-free languages. Let $\mathcal{CFL}_\Sigma$ be the class of context-free languages over Σ, and let *μ-regular algebra* (over Σ) be the equational theory of $\mathcal{CFL}_\Sigma$. We are interested in subtheories of μ-regular algebra that can be axiomatized in the language of μ-regular expressions. In the rest of this section, we will consider a very basic theory of this kind, fixing μ to be a least fixed point operator.

We want to look at some ways to define * in terms of μ, and for that reason *we will not use * in μ-regular expressions henceforth*, but only $+$, $\cdot$, 0, 1, and μ. The models of our theory are certain structures $\mathcal{M} = (M, +, 0, \cdot, 1, \mu)$, where $(M, +, 0, \cdot, 1)$ is a first-order structure, and μ a functional on M that associates to every definable function $f : M \to M$ an element $\mu(f)$ of M, the least pre-fixed point of f.

Definition 2.2 The theory of *Kleene Algebra with least fixed points*, *KAF*, consists of (i) the theory of idempotent semirings (as above), and (ii) the following schemata of least pre-fixed points: for all μ-regular expressions r,

$$r[\mu x.r/x] \le \mu x.r, \qquad (6) \qquad\qquad \text{and} \qquad\qquad \forall x(r \le x \to \mu x.r \le x). \qquad (7)$$

A more accurate name would be *idempotent semiring with least fixed points*, in particular since we dropped *. The expected monotonicity and fixed-point properties follow easily from the axioms for μ:

Proposition 2.3 *For all μ-regular expressions r,s, and variables $y \not\equiv x \not\equiv z$,*

$$\begin{aligned} KAF &\ \vdash\ \forall x(s \le r) \to \mu x.s \le \mu x.r, && (8)\\ KAF &\ \vdash\ \forall y \forall z\ (y \le z \to \mu x.r[y/v] \le \mu x.r[z/v]), && (9)\\ KAF &\ \vdash\ \mu x.r = r[\mu x.r/x]. && (10) \end{aligned}$$

Proof To show (8), from $\forall x(s \le r)$ we first get $s[\mu x.r/x] \le r[\mu x.r/x] \le \mu x.r$, using (6) for the second step. With (7), this then gives $\mu x.s \le \mu x.r$. To show (9) for $r(x, v)$, note that by induction we may assume $y \le z \to \forall x(r(x, y) \le r(x, z))$, using monotonicity of the regular operations for μ-free expressions r. By (8), we get $\mu x.r(x, y) \le \mu x.r(x, z)$. Finally, for (10), from axiom (6) we get $r[r[\mu x.r/x]/x] \le r[\mu x.r/x]$ by monotonicity, and by axiom (7) this implies $\mu x.r \le r[\mu x.r/x]$. □

Let us now see whether we can define Kleene's iteration * in *KAF*. At first sight, Kozen's axioms for * are instances of our μ-axioms for $r(a, b, x) := (b + ax)$,

$$b + a \cdot \mu x(b + ax) \le \mu x(b + ax), \qquad \text{and} \qquad \forall x(b + ax \le x \to \mu x(b + ax) \le x).$$

Writing a^*b for $\mu x(b + ax)$ these are just the first (with $b = 1$) and second part of (4). Similarly, we obtain (5) by taking $\mu x(b + xa)$ for ba^*.

However, this does not give *one* definition for a^*, by taking $b = 1$, but two of them: the *right iteration* $\mu x(1 + ax)$ and the *left iteration* $\mu x(1 + xa)$ of a, respectively. Since $\cdot$ need not be commutative, in some models of KAF it can make a difference whether we iterate a to the left or to the right. But in the standard interpretations of the Introduction, both iterations of a coincide, and in fact are equal not only to the *both-sided iteration* $\mu x(1 + ax + xa)$ of a, but also to $\mu x(1 + a + xx)$, the *reflexive transitive closure* of a. It seems that in KAF, we can only prove part of this:

Proposition 2.4 $KAF \vdash \mu x(1{+}ax){+}\mu x(1{+}xa) \leq \mu x(1 + ax + xa) \leq \mu x(1 + a + xx)$.

Proof The first inequation is obvious. For the second, let $\bar{x}$ be $\mu x(1 + a + xx)$. Then (i) $1 \leq \bar{x}$, (ii) $a \leq \bar{x}$, and (iii) $\bar{x}\bar{x} \leq \bar{x}$. By monotonicity of $+$ and $\cdot$, we get

$$1 + a\bar{x} + \bar{x}a \leq_{(ii)} 1 + \bar{x}\bar{x} + \bar{x}\bar{x} \leq_{(iii)} 1 + \bar{x} + \bar{x} \leq_{(i)} \bar{x} + \bar{x} + \bar{x} = \bar{x}.$$

By the minimality axiom (7), this implies $\mu x(1 + ax + xa) \leq \bar{x}$. □

We note that Conway has given a finite nonstandard model of regular algebra where a^* is *not* the reflexive transitive closure of a. (Such models are excluded by ACT and KA.) To understand better why the various definitions of a^* coincide in the language and in the relation interpretation, we now look at a specific class of models of KAF.

3 Continuous models of KAF

As is well-known, an n-ary function $f : M^n \to M$ on a complete partial order $(M, 0, \leq)$ is *Scott-continuous* if and only if for all i and all parameters $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$,

$$f(b_1, \ldots, b_{i-1}, \bigsqcup D, b_{i+1}, \ldots, b_n) = \bigsqcup_{d \in D} f(b_1, \ldots, b_{i-1}, d, b_{i+1}, \ldots, b_n),$$

for each directed set $D \subseteq M$. Moreover, every continuous function $f : M \to M$ has a least fixed-point $\mu x.f(x)$, which, by Kleene's fixed point theorem, is

$$\mu x.f(x) := \bigsqcup_{n \in \omega} f_n(0), \qquad \text{where } f_0(x) = 0,\ f_{k+1}(x) := f(f_k(x)). \tag{11}$$

Definition 3.1 An idempotent semiring $(M, +, 0, \cdot, 1)$ is *continuous*, if the partial ordering $\leq$ induced by $+$ makes $(M, \leq, 0)$ a complete partial order, and $+$ and $\cdot$ are continuous functions with respect to the Scott topology on $(M, 0, \leq)$ and its cartesian product. A *continuous model of* KAF is a model $\mathcal{M}$ of KAF whose underlying semiring is continuous.

Proposition 3.2 *The family $\mathcal{L}_\Sigma$ of all languages over an alphabet Σ and the family $\mathcal{R}_K$ of all binary relations over a set K each form a continuous model of KAF.*

The proof is obvious and hence omitted. But note that although the class of context-free languages has enough fixed points to form a model of KAF, it is *not* a continuous model, because its partial ordering is incomplete (in fact, not even closed under unions of increasing chains).

The next lemma shows that the various definitions for a^* coincide not only in the standard interpretations, but in all (substructures of) continuous models of KAF.

Lemma 3.3 *If $\mathcal{M} \models KAF$ is continuous, $\mathcal{M} \models \forall a.\ \mu x(1 + ax) = \mu x(1 + a + xx)$.*

Proof By Proposition 2.4, it is sufficient to prove $\mathcal{M} \models \forall a.\ \mu x(1 + a + xx) \leq \mu x(1 + ax)$. Let $\bar{x}$ be $\mu x(1 + ax)$, whence we have $1 + a\bar{x} \leq \bar{x}$. From $1 + a\bar{x} \leq \bar{x}$ we get (i) $1 \leq \bar{x}$, and then (ii) $a \leq \bar{x}$ using $a = a \cdot 1 \leq a \cdot \bar{x} \leq \bar{x}$. To show (iii) $\bar{x}\bar{x} \leq \bar{x}$, define $x_0 := 0$, $x_{n+1} := r^{\mathcal{M}}[x_n/x]$, where $r = (1 + ax)$. By monotonicity of $+$ and $\cdot$, $x_n \leq x_{n+1}$, and by completeness of the partial order, there is a least upper bound $\sqcup_{n\in\omega} x_n \in M$ for $\{x_n \mid n \in \omega\}$. Since $\lambda x.r$ is continuous on M by the continuity of $+$ and $\cdot$, from Kleene's fixed-point theorem (11) we get $\mu x.r = \sqcup_{n\in\omega} x_n$. Finally, we need $x_n\bar{x} \leq \bar{x}$ for all n: this is clear for $n = 0$, and by induction,

$$x_{n+1}\bar{x} = (1 + ax_n)\bar{x} = \bar{x} + ax_n\bar{x} \leq \bar{x} + a\bar{x} \leq \bar{x},$$

using $a\bar{x} \leq 1 + a\bar{x} \leq \bar{x}$ in the last step. Hence, (iii) holds, since

$$\bar{x}\bar{x} = (\sqcup_{n\in\omega} x_n)\bar{x} = \sqcup_{n\in\omega}(x_n\bar{x}) \leq \sqcup_{n\in\omega}\bar{x} = \bar{x}.$$

From (i) - (iii) it follows that $\mu x(1 + a + xx) \leq \bar{x}$, by minimality. □

Theorem 3.4 *Every continuous idempotent semiring $(M, +, 0, \cdot, 1)$ can uniquely be expanded to a continuous model $\mathcal{M} = (M, +, 0, \cdot, 1, \mu)$ of KAF.*

To show this, note that least fixed points of continuous functions exist by (11), and so it remains to note that nesting least fixed-points does not take us beyond the continuous functions:

Lemma 3.5 *Let $\mathcal{M}$ be a continuous idempotent semiring, and $r(x, y_1, \ldots, y_n)$ a continuous function from M^{n+1} to M. The function*

$$\lambda(a_1, \ldots, a_n)\ \mu b.r(b, a_1, \ldots, a_n)\ :\ M^n \to M$$

that picks the least fixed point of $\lambda b.r(b, a_1, \ldots, a_n)$ according to (11), is a continuous function on M^n.

Proof It is sufficient to show continuity in each dimension, so let r be $r(x, y)$, and $Y \subseteq M$ a directed set. To show

$$\mu x.r(x, \bigsqcup Y) = \bigsqcup_{y\in Y} \mu x.r(x, y),$$

we write $\bar{x} := \mu x.r(x, \bigsqcup Y)$, $x_y := \mu x.r(x, y)$ for $y \in Y$, and $x_Y := \bigsqcup_{y \in Y} x_y$.

Claim 1: $\bar{x} \geq x_Y$. By definition of $\bar{x}$ and monotonicity of r in y,

$$\bar{x} \geq r(\bar{x}, \bigsqcup Y) \geq \bigsqcup_{y \in Y} r(\bar{x}, y) \geq r(\bar{x}, y)$$

for each $y \in Y$, and hence $\bar{x} \geq \mu x.r(x, y) = x_y$ for each $y \in Y$. Since Y is directed, so is $\{r(\bar{x}, y) \mid y \in Y\}$, and taking the sup gives $\bar{x} \geq \bigsqcup_{y \in Y} x_y = x_Y$.

Claim 2: $\bar{x} \leq x_Y$. Note that

$$\begin{array}{rclcl} r(x_Y, \bigsqcup Y) & = & \bigsqcup_{y \in Y} r(x_Y, y) & = & \bigsqcup_{y \in Y} \bigsqcup_{z \in Y} r(x_z, y) \\ & = & \bigsqcup_{y \in Y, z \in Y} r(x_z, y) & = & \bigsqcup_{y \in Y} r(x_y, y) \\ & \leq & \bigsqcup_{y \in Y} x_y & = & x_Y, \end{array}$$

and hence $\bar{x} \leq x_Y$ by minimization. □

Conway[2] has studied several notions of Kleene algebra, and we next relate continuous models of *KAF* to his Standard Kleene Algebras, which are defined in terms of an infinitary summation Σ.

Definition 3.6 (Conway[2], Chapter 3) A *Standard Kleene Algebra*, or *$\mathcal{S}$-algebra*, is a structure $(M, \sum, 0, \cdot, 1, ^*)$ satisfying the following conditions, for arbitrary index sets I, J, J_i $(i \in I)$ and elements $e_i \in M$, using $e^0 = 1$, $e^{n+1} = e^n \cdot e$:

$$\begin{array}{rclcrcl} \sum_{i \in \emptyset} e_i & = & 0 & \qquad & e \cdot 1 = & e & = 1 \cdot e \\ \sum_{i \in I} (\sum_{j \in J_i} e_j) & = & \sum_{j \in \cup_{i \in I} J_i} e_j & & (e_1 \cdot e_2) \cdot e_3 & = & e_1 \cdot (e_2 \cdot e_3) \\ (\sum_{i \in I} e_i) \cdot (\sum_{j \in J} e_j) & = & \sum_{(i,j) \in I \times J} e_i \cdot e_j & & e^* & = & \sum_{n \in \omega} e^n. \end{array}$$

The binary addition is defined by $e_1 + e_2 := \sum_{i \in \{1,2\}} e_i$. The properties of $\sum$ make $(M, +, 0, \cdot, 1)$ an idempotent semiring, and hence $e_1 \leq e_2 :\leftrightarrow e_1 + e_2 = e_2$ induces a partial order $\leq$ on M. According to Kozen, "$\mathcal{S}$-algebras are defined in terms of an infinitary summation operator $\sum$, whose sole purpose, it seems, is to define *". Actually, there is more to $\sum$ than defining *: by the properties of $\sum$, M is a complete semi-lattice with respect to $\leq$, and the operations $+$ and $\cdot$ are Scott-continuous with respect to $\leq$.

Proposition 3.7 *Every $\mathcal{S}$-algebra $\mathcal{M}$ can be expanded to a continuous model of KAF, by defining a μ-operator via*

$$(\mu x.r)^{\mathcal{M}} := \sum_{n \in \omega} r_n^{\mathcal{M}} \qquad \textit{with } r_0^{\mathcal{M}} := 0,\ r_{n+1}^{\mathcal{M}} := r^{\mathcal{M}}[r_n^{\mathcal{M}}/x].$$

The structure $(\mathcal{M}, \mu)$ satisfies $\forall a(\mu x(1 + ax) = a^*)$.

Proof By Kleene's fixed point theorem (11), the defined μ makes $\mu x.r$ the least fixed point of $\lambda x.r$ on $\mathcal{M}$. This implies that the μ-axioms of *KAF* are satisfied. Continuity and the semiring properties follow from Conway's axioms about $\sum$. Finally, $a^* = \sum_{n \in \omega} a^n = \mu x(1 + ax)^{\mathcal{M}}$. □

Conway[2] (Chapter 4, Theorem 6) shows that $\mathcal{L}_\Sigma$, the family of all formal languages over the alphabet Σ, is the free $\mathcal{S}$-algebra in the generators Σ. We now give a similar characterization for $\mathcal{CFL}_\Sigma$ and the class of substructures of continuous models of KAF. The construction needs a bit more care than the one for $\mathcal{S}$-algebras, since models of KAF need not even be closed under suprema of monotone sequences, as $\mathcal{CFL}_\Sigma$ shows.

Definition 3.8 A subset $A \subseteq M$ of a structure $\mathcal{M}$ for the language $\{+, 0, \cdot, 1, \mu\}$ is a *set of generators for* M if every element of M is the value $r^{\mathcal{M}}[a_1, \ldots, a_n]$ of a pure μ-regular expression $r(x_1, \ldots, x_n)$ with parameters $a_1, \ldots, a_n$ from A. If $\mathcal{C}$ is a class of structures, $\mathcal{M} \in \mathcal{C}$ is *free for* $\mathcal{C}$ in its set $A \subseteq M$ of generators, if for all elements $a_1, \ldots, a_n$ of A and all μ-regular expressions s and t,

$$\mathcal{M} \models (s = t)[a_1/x_1, \ldots, a_n/x_n] \quad \Rightarrow \quad \mathcal{C} \models \forall x_1 \ldots \forall x_n (s = t).$$

This means that in a free structure for a class of models only those relations between generators hold that are universally valid in the class.

Theorem 3.9 *The algebra $\mathcal{CFL}_\Sigma$ of all context-free languages over the finite alphabet Σ is free for the class of substructures of continuous models of KAF with $|\Sigma|$ many generators.*

Proof By Theorem 1.1, $\mathcal{CFL}_\Sigma$ is just the substructure of $\mathcal{L}_\Sigma$ that consists of the μ-regularly definable languages (that is, definable without parameters other than the $a \in \Sigma$). As $\mathcal{L}_\Sigma$ is an $\mathcal{S}$-algebra, by Proposition 3.7 we know that $\mathcal{CFL}_\Sigma$ is a substructure of a continuous model of KAF. We assume $\Sigma = \{a_1, \ldots, a_n\}$ and write $\mathcal{CF}$ instead of $\mathcal{CFL}_\Sigma$.

Let r and s be pure regular expressions such that $\mathcal{CF} \models (r = s)[\{a_1\}, \ldots, \{a_n\}]$, and $\mathcal{M} = (M, +, 0, \cdot, 1, \mu)$ be a continuous model of KAF. To show $\mathcal{M} \models \forall x_1 \ldots x_n . r = s$, let $b_1, \ldots, b_n$ be elements of $\mathcal{M}$. It is sufficient to present a homomorphism from $\mathcal{CF}$ to $\mathcal{M}$, mapping the atoms $\{a_i\}$ to the b_i.

Define $^- : \mathcal{CF} \to \mathcal{M}$ by putting $\overline{L} := \bigsqcup\{\sum_{w \in E} \overline{w} \mid E \subseteq L \text{ is finite}\}$, for context-free languages $L \subseteq \Sigma^*$, where $\overline{v \cdot w} := \overline{v} \cdot \overline{w}$, $\overline{a_i} := b_i$, and $\overline{\epsilon} := 1$. We leave it to the reader to check that for all languages L_1, L_2,
Claim 1: $\overline{L_1 + L_2} = \overline{L_1} + \overline{L_2}$, $\quad \overline{L_1 \cdot L_2} = \overline{L_1} \cdot \overline{L_2}$, $\quad$ and $\quad \overline{\emptyset} = 0^{\mathcal{M}}$, $\quad \overline{\{\epsilon\}} = 1^{\mathcal{M}}$.

To see that $^-$ is a homomorphism, we show the following
Claim 2: For every μ-regular expression $r(x_0, \ldots, x_n)$ and all languages $L_0, \ldots, L_n$ in $\mathcal{CFL}_\Sigma$:

(i) $\overline{r^{\mathcal{CF}}[L_0, \ldots, L_n]} = r^{\mathcal{M}}[\overline{L}_0, \ldots, \overline{L}_n]$,

(ii) $\overline{r_k^{\mathcal{CF}}[L_1, \ldots, L_n]} = r_k^{\mathcal{M}}[\overline{L}_1, \ldots, \overline{L}_n]$, for all $k \in \omega$, where for all models $\mathcal{A}$ and $L_i \in A$,

$$r_0^{\mathcal{A}}[L_1, \ldots, L_n] := 0^{\mathcal{A}} \text{ and } r_{k+1}^{\mathcal{A}}[L_1, \ldots, L_n] := r^{\mathcal{A}}[r_k^{\mathcal{A}}[L_1, \ldots, L_n], \ldots, L_1, \ldots, L_n].$$

Proof by induction on the nesting depth of μ's in r. If r has μ-depth 0, (i) is clear by Claim 1, as is (ii) for $k = 0$. By induction, one has

$$\begin{array}{rcll}\overline{r_{k+1}^{\mathcal{CF}}[L_1,\dots,L_n]} & = & \overline{r^{\mathcal{CF}}[r_k^{\mathcal{CF}}[L_1,\dots,L_n],L_1,\dots,L_n]} & \\ & = & r^{\mathcal{M}}[\overline{r_k^{\mathcal{CF}}[L_1,\dots,L_n]},\overline{L}_1,\dots,\overline{L}_n] & \text{(by (i))} \\ & = & r^{\mathcal{M}}[r_k^{\mathcal{M}}[\overline{L}_1,\dots,\overline{L}_n],\overline{L}_1,\dots,\overline{L}_n] & \text{(by induction)} \\ & = & r_{k+1}^{\mathcal{M}}[\overline{L}_1,\dots,\overline{L}_n], & \end{array}$$

and hence (ii) is shown. For μ-depth $k + 1$, consider $\mu x.r$ where $r(x, x_0, \dots, x_n)$ has μ-depth at most k. To see part (i), we calculate

$$\begin{array}{rcl}\overline{(\mu x.r)^{\mathcal{CF}}[L_0,\dots,L_n]} & = & \overline{\bigcup_{k\in\omega} r_k^{\mathcal{CF}}[L_0,\dots,L_n]} \\ & = & \bigsqcup\{\sum_{w\in E}\overline{w} \mid E \subseteq \bigcup_{k\in\omega} r_k^{\mathcal{CF}}[L_0,\dots,L_n] \text{ finite}\} \\ & = & \bigsqcup\bigcup_{k\in\omega}\{\sum_{w\in E}\overline{w} \mid E \subseteq r_k^{\mathcal{CF}}[L_0,\dots,L_n] \text{ finite}\} \\ (*) & = & \bigsqcup_{k\in\omega}\overline{r_k^{\mathcal{CF}}[L_0,\dots,L_n]} \\ & = & \bigsqcup_{k\in\omega}(r_k^{\mathcal{M}}[\overline{L}_0,\dots,\overline{L}_n]) \quad \text{(by (ii) for } r\text{)} \\ & = & (\mu x.r)^{\mathcal{M}}[\overline{L}_0,\dots,\overline{L}_n],\end{array}$$

using in (*) that $\bigsqcup(\bigcup_{i\in\omega} M_i) = \bigsqcup_{i\in\omega}(\sqcup M_i)$ for ascending chains $M_0 \subseteq M_1 \subseteq \dots \subseteq M$. Part (ii) follows from (i) for $\mu x.r$, exactly as shown for μ-depth 0 above.

By induction on pure μ-regular expressions, these claims yield

$$\overline{r^{\mathcal{CF}}[L_1,\dots,L_n]} = r^{\mathcal{M}}[\overline{L}_1,\dots,\overline{L}_n] \tag{12}$$

for every $r(x_1, \dots, x_n)$; using $L_i = \{a_i\}$ shows that $^-$ is a homomorphism.

If $\mathcal{N}$ is the substructure of $\mathcal{M}$ that is μ-regularly generated by the elements b_i, this homomorphism is onto $\mathcal{N}$. If, in addition, $\mathcal{N}$ is a free structure for the class of substructures of continuous models of KAF, then $^-$ is an isomorphism. □

4 Kleene algebras with recursion

In Section 2 we have seen that Kozen's axiom for * could *almost* be seen as instances of the least fixed point properties of KAF. The problem was that $\mu x(b + ax)$ and $\mu x(b + xa)$, intended to represent $a^* \cdot b$ and $b \cdot a^*$, do not agree for $b = 1$ to give one candidate for a^*.

Reading the $\mu x(b + xa)$ as a recursive program, it is clear that in terminating executions essentially we *first* do b *and then* repeatedly do a, i.e. the finite behaviour of recursive programs satisfies $\mu x(b + xa) = b \cdot \mu x(1 + ax)$. The following makes this link between recursion and iteration explicit.

Definition 4.1 The theory of *Kleene Algebra with recursion*, KAR, is the extension of the above theory KAF by the following assumptions, for all a and b:

$$b \cdot \mu x(1 + ax) \;\leq\; \mu x(b + xa) \tag{13}$$

$$\mu x(1 + xa) \cdot b \;\leq\; \mu x(b + ax) \tag{14}$$

It seems plausible that these axioms are independent of KAF and of each other, because the minimality conditions in axioms (4) and (5) are independent over KA, as shown by Kozen[7].

Proposition 4.2 *In KAF, (13) and (14) together are equivalent to*

$$\mu x(b + xa) = b \cdot \mu x(1 + ax) \tag{15}$$
$$\mu x(b + ax) = \mu x(1 + xa) \cdot b \tag{16}$$

Proof To see that (15) and (16) follow from the axioms, first note that by taking $b = 1$ in (13) and (14), we obtain

$$KAR \vdash \mu x(1 + ax) = \mu x(1 + xa). \tag{17}$$

The missing inequations can now be obtained by substituting (17) into the following:

$$KAF \vdash \mu x(b + xa) \leq b \cdot \mu x(1 + xa) \tag{18}$$
$$KAF \vdash \mu x(b + ax) \leq \mu x(1 + ax) \cdot b \tag{19}$$

To show (19), we use $\bar{x} := \mu x(1 + ax)$. By definition, we have $1 \leq \bar{x}$ and $a\bar{x} \leq \bar{x}$, and thus

$$b + a\bar{x}b \leq b + \bar{x}b \leq (1 + \bar{x})b \leq \bar{x}b.$$

This implies $\mu x(b + ax) \leq \bar{x}b$ by minimality. By symmetry, we also have (18). □

We are now ready to see that in KAR all of the previously discussed iterations of a coincide.

Lemma 4.3 $KAR \vdash \mu x(1+ax) = \mu x(1+xa) = \mu x(1+ax+xa) = \mu x(1+a+xx)$.

Proof The claim reduces by (17) and Proposition 2.4 to $\mu x(1+a+xx) \leq \mu x(1+ax)$. Since obviously $1 + a \leq \mu x(1 + ax)$, we only have to show that

$$\mu x(1 + ax) \cdot \mu x(1 + ax) \leq \mu x(1 + ax), \tag{20}$$

and then use minimality of $\mu x(1 + a + xx)$. We can now mimick an argument of Pratt[12], abbreviating $\mu x(b + ax)$ by $[a^*, b]$ for clarity. By the μ-axioms of KAF, we have

$$b + a[a^*, b] \leq [a^*, b], \tag{21}$$
$$b + ax \leq x \rightarrow [a^*, b] \leq x. \tag{22}$$

From (21) and $[a^*, b] \leq [a^*, b]$ we get $[a^*, b] + a[a^*, b] \leq [a^*, b]$, hence $[a^*, [a^*, b]] \leq [a^*, b]$ by (22). Equations (17) and (14) yield $[a^*, 1] \cdot b \leq [a^*, b]$, and by instantiating b to $[a^*, b]$ we get

$$[a^*, 1] \cdot [a^*, b] \leq [a^*, [a^*, b]] \leq [a^*, b].$$

Taking $b = 1$ here gives transitivity $[a^*, 1] \cdot [a^*, 1] \leq [a^*, 1]$ of $[a^*, 1]$, which is (20). □

From equations (16) and (17) and the discussion of Kozen's axioms in Section 2, we conclude that all the μ-definable iterations above have the properties of Kleene's iteration operator *:

Corollary 4.4 *Under the translation $a^* := \mu x(1 + ax)$, KA is a subtheory of KAR.*

By Kozen's completeness theorem for *KA*, it follows that *every* equation between regular expressions that is valid in the language interpretation can be proven in *KAR*.

It is natural to ask whether *KAR* is complete for the equational theory of context-free languages, i.e. whether every equation between *μ-regular* expressions valid in the language interpretation is provable in *KAR*. This is not the case: the equivalence between context-free grammars is not recursively enumerable (combine Corollary 1 with the proof of Theorem 8.11 of Hopcroft/Ullman[5], p. 192 and p. 203). Since there is an effective translation between context-free grammars and μ-regular expressions (in a proof of Theorem 1.1), the equational theory of context-free languages in terms of μ-regular expressions is not axiomatizable at all.

Clearly, the standard interpretations give models not only of *KAF*, but also of *KAR*. It follows from Theorem 5.3 below that every continuous model of *KAF* is a model of *KAR*.

5 Elimination of left-recursion

The axioms of *KAR* are special instances of S. Greibach's trick for eliminating left-recursive grammar rules by right-recursive ones and vice versa. We will now express the core of Greibach's method as an equation between μ-regular expressions that appears to be independent of *KAR*, but is valid in all continuous models of *KAF*.

Definition 5.1 The theory of *Kleene Algebra with Greibach's inequations for recursion*, *KAG*, is the extension of the theory *KAF* by the following axiom schemata: for all μ-regular expressions r and s with y not free in r, add the universal closures of

$$\mu x(s \cdot \mu y(1 + ry)) \leq \mu x(s + xr) \tag{23}$$
$$\mu x(\mu y(1 + yr) \cdot s) \leq \mu x(s + rx) \tag{24}$$

Note that *KAG* extends *KAR*: if x is not free in s and r, the μx on the left hand side of (23) and (24) can be dropped; for variables r and s this gives axioms (15) and (16) of *KAR*.

Proposition 5.2 *In KAF, the schemata (23) and (24) together are equivalent to the following two schemata for μ-regular expressions r and s with y not free in r:*

$$\mu x(s \cdot \mu y(1 + ry)) = \mu x(s + xr), \tag{25}$$
$$\mu x(\mu y(1 + yr) \cdot s) = \mu x(s + rx). \tag{26}$$

Proof By the above remark, it is sufficient to show

$$KAR \vdash \mu x(xr + s) \leq \mu x(s \cdot \mu y(1 + ry)). \tag{27}$$

Let $\bar{x}$ be $\mu x(s(x) \cdot \mu y(1 + r(x) \cdot y))$, suppressing other free variables in the notation. We prove

$$\bar{x} \cdot r(\bar{x}) + s(\bar{x}) \leq \bar{x}, \tag{28}$$

which implies (27) by minimization. Let $\bar{y}$ be $\mu y(1 + r(\bar{x}) \cdot y)$. By properties of $\bar{x}$ and $\bar{y}$, we get $s(\bar{x}) \cdot (1 + r(\bar{x}) \cdot \bar{y}) \leq s(\bar{x}) \cdot \bar{y} \leq \bar{x}$, hence $s(\bar{x}) \leq \bar{x}$ and $s(\bar{x}) \cdot r(\bar{x}) \cdot \bar{y} \leq \bar{x}$. It remains to show $\bar{x} \cdot r(\bar{x}) \leq \bar{x}$. From (15), (16) and (17) we have

$$KAR \vdash \forall a.\ \mu y(1 + ay) \cdot a = \mu y(a + ay) = a \cdot \mu y(1 + ay).$$

Thus, $\bar{y} \cdot r(\bar{x}) = \mu y(1 + r(\bar{x}) \cdot y) \cdot r(\bar{x}) = r(\bar{x}) \cdot \mu y(1 + r(\bar{x}) \cdot y) = r(\bar{x}) \cdot \bar{y}$, and hence (28) follows by $\bar{x} \cdot r(\bar{x}) = s(\bar{x}) \cdot \bar{y} \cdot r(\bar{x}) = s(\bar{x}) \cdot r(\bar{x}) \cdot \bar{y} \leq \bar{x}$. □

To explain why we call (23) and (24) *Greibach's axioms*, let us recall S. Greibach's[3] way to eliminate left recursive rules from a context-free grammar (c.f. [5, 4]). Suppose

$$A = Av_1 + \cdots + Av_n + w_1 + \cdots + w_m$$

combines all the A-rules of the grammar, where the v_i and w_j are concatenations of grammar symbols, and no w_j begins with the variable A. Using a fresh variable B, the rules corresponding to the above equation are replaced by those corresponding to the equations

$$\begin{array}{rcl} A & = & w_1 + \cdots + w_m + w_1 B + \cdots + w_m B, \\ B & = & v_1 + \cdots + v_n + v_1 B + \cdots + v_n B. \end{array}$$

Writing r for $(v_1 + \cdots + v_m)$ and s for $(w_1 + \cdots + w_m)$, we just have replaced the equation $A = Ar + s$ by the equations $A = s + sB$ and $B = r + rB$. Since we are dealing with *least* solutions, we have in fact replaced $\mu A(Ar + s)$ by $\mu A(s + s \cdot \mu B(r + rB))$, i.e. we used

$$\mu A(s + Ar) = \mu A(s + s \cdot \mu B(r + rB)). \tag{29}$$

As B was fresh, in KAR we can reformulate the right hand side as

$$\begin{array}{rcl} \mu A(s + s \cdot \mu B(r + rB)) & = & \mu A(s \cdot (1 + \mu B(r + rB))) \\ & = & \mu A(s \cdot (1 + r \cdot \mu B(1 + rB))) \\ & = & \mu A(s \cdot \mu B(1 + rB)), \end{array}$$

and (29) becomes $\mu A(s + Ar) = \mu A(s \cdot \mu B(1 + rB))$, an instance of (25)[3]. Of course, for suitable s only, in particular those corresponding to the above constraint that no w_j begins with A, can we conclude that $\mu A(s \cdot \mu B(1 + rB))$ is not left-recursive (with respect to μA).

It is well known that for *fixed* sets r and s of words, the least solution of $x = xr + s$ is sr^*. This is just the content of axiom (15) of KAR, using the right-recursive definition $\mu x(1 + ax)$ for the iteration a^*.[4]

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[3]This corresponds to the replacement of the original grammar rules for A by the new equations

$$A = w_1 B + \cdots + w_m B, \qquad B = 1 + v_1 B + \cdots + v_n B.$$

These are simpler than the standard ones given above, but contain an undesired 'ϵ-rule' in $B = 1 + rB$.

[4]Note that $\mu x(b + ax) = a^* \cdot b$ can be read as *'tail recursion is* implementable by *iteration'*.

Greibach's 'elimination of left recursion' is based on a more general fact, expressed in (25): the least solution of $x = xr + s$ is *the least solution of* $x = sr^*$, even if r and s μ-regularly *depend* on x. We finally prove that this is a true identity not only in the language interpretation, but in all continuous models of KAF:

Theorem 5.3 *Every continuous model of KAF is a model of KAG.*

Proof Let $\mathcal{M}$ be a continuous model of KAF. By symmetry, it is sufficient to show that $\mathcal{M}$ satisfies (23). Let $\bar{x}$ be $\mu x(x \cdot r(x) + s(x))$ in $\mathcal{M}$, under a given assignment for free variables that are not explicitly mentioned. We will show that $s(\bar{x}) \cdot \mu y(1 + r(\bar{x})y) \leq \bar{x}$, which gives (23) by minimization. Let $\bar{y} := \mu y(1 + r(\bar{x})y)$ in $\mathcal{M}$. Since $\bar{x} \cdot r(\bar{x}) \leq \bar{x}$ and $s(\bar{x}) \leq \bar{x}$ by the choice of $\bar{x}$, we get $s(\bar{x}) \cdot \bar{y} \leq \bar{x}\bar{y}$. To show the remaining inequation $\bar{x}\bar{y} \leq \bar{x}$, note that since $\mathcal{M}$ is continuous, $\bar{y} = \sqcup_{n\in\omega} y_n$ with $y_0 := 1 + r(\bar{x}) \cdot 0 = 1$, $y_{n+1} := 1 + r(\bar{x}) \cdot y_n$. From $\bar{x}y_0 \leq \bar{x}$ we get $\bar{x}y_{n+1} = \bar{x} + \bar{x} \cdot r(\bar{x}) \cdot y_n \leq \bar{x} + \bar{x}y_n \leq \bar{x}$ by induction, so $\bar{x}\bar{y} = \sqcup_{n\in\omega} \bar{x}y_n \leq \bar{x}$ by continuity. □

6 Open problems

We have demonstrated that μ-regular expressions can be useful to study questions about equality and subsumption between context-free languages in an algebraic and logical manner. The two main arguments in this exercise have been (i) minimality of least fixed points on a continuous idempotent semiring and (ii) induction on the number of iterations of a definable monotone function on this ring.

The axiom systems $KAF \subseteq KAR \subseteq KAG$ separate the basic properties of least fixed points from aspects of continuity that can be expressed as equations between μ-regular expressions. But we have left open all non-trivial questions, for example:

(i) Is KAF strictly weaker than KAR, and KAR strictly weaker than KAG?

(ii) Is KAF, or KAG relative to KAF, finitely axiomatizable?

(iii) Let $\mathcal{K}$ be a model of KAR. Can the algebra $\mathcal{K}_n$ of $n \times n$-matrices over $\mathcal{K}$ be expanded to a model of KAR by adding an appropriate μ? Is the same true for KAG? (C.f. Conway[2] for $\mathcal{S}$-algebras and Kozen[8] for models of KA.)

(iv) Are there natural equations between μ-regular expressions that are valid in all continuous models of KAF, but go beyond KAG? Good candidates are those equations that arise by transforming a simultaneously regular definition into different μ-regular ones.

(v) Is the equational theory of *linear* languages, i.e. linear μ-regular expressions, more tractable than the general case?

Note that in order to show strictness in (i), we need non-continuous models in which the definable functions have least fixed points.

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