Fast String Correction with Levenshtein-Automata

Klaus U. Schulz
CIS
University of Munich
schulz@cis.uni-muenchen.de

Stoyan Mihov
Linguistic Modelling Laboratory
LPDP – Bulgarian Academy of Sciences
stoyan@lml.bas.bg

Abstract

The Levenshtein-distance between two words is the minimal number of insertions, deletions or substitutions that are needed to transform one word into the other. Levenshtein-automata of degree n for a word W are defined as finite state automata that regognize the set of all words V where the Levenshtein-distance between V and W does not exceed n. We show how to compute, for any fixed bound n and any input word W, a deterministic Levenshtein-automaton of degree n for W in time linear in the length of W. Given an electronic dictionary that is implemented in the form of a trie or a finite state automaton, the Levenshtein-automaton for W can be used to control search in the lexicon in such a way that exactly the lexical words V are generated where the Levenshtein-distance between V and W does not exceed the given bound. This leads to a very fast method for correcting corrupted input words of unrestricted text using large electronic dictionaries. We then introduce a second method that avoids the explicit computation of Levenshtein-automata and leads to even improved efficiency. We also describe how to extend both methods to variants of the Levenshtein-distance where further primitive edit operations (transpositions, merges and splits) may be used.

Keywords: Spelling correction, Levenshtein-distance, optical character recognition, electronic dictionaries.

Contents

Chapter 1

Introduction and Motivation

The problem of how to find good correction candidates for a garbled input word is important for many fundamental applications, including spelling correction, speech recognition, OCR-recognition, as well as internet and bibliographic search. Due to its relevance the problem has been considered by many authors (e.g., [Bla60, RE71, Ull77, AFW83, SHC83, Sri85, TIAY90, Kuk92, ZD95, DHH$^+$97]). Most contributions suggest methods for correcting isolated words of a text.[1] Since purely statistical methods cannot offer sufficient correction accuracy, modern approaches are generally built on top of lexical techniques.

If an electronic dictionary is available that covers the possible input words, a simple procedure may be used for detecting and correcting errors. Given an input word W, it is first checked if the word is in the dictionary. In the negative case, the words of the dictionary that are most similar to W are suggested as correction candidates. If necessary, appropriate statistical data can be used for refinement of ranking. Similarity between two words can be measured in several ways. Most useful are (dis)similarity measures based on variants of the Levenshtein-distance [Lev66, WF74, WBR95, SKS96, OL97] or on n-gram distances [AFW83, Ukk92, KST92, KST94]. In this paper, we take the Levenshtein-distance as a basis.

The standard algorithm for computing the Levenshtein-distance between two words by Wagner and Fisher [WF74] uses a dynamic programming scheme that leads to quadratic time complexity. Even with more sophisticated algorithms (cf. [Ukk85]) it is not realistic to compute the Levenshtein-distance between the input word W and each of the words in the dictionary, already for dictionaries of a modest size. The problem becomes even more serious when using dictionaries for highly inflectional or agglutinating languages (e.g., Russian, German, Turkish, Finnish, Hungarian), dictionaries for languages that allow for composition of nouns (german), or multi-lingual dictionaries. In these cases, dictionaries may contain several millions of entries. The problem arises of how to compute the lexical Levenshtein-neighbours of a garbled input word while respecting the efficiency constraints that arise from realistic industrial applications.

Several solutions have been proposed for fast selection of possible corrections. Often the dictionary is offline partitioned using a similarity key [Sin90, Kuk92, dBdBT95, ZD95], or it is enriched with a special index structure [OM88, KST92,

[1] Some more recent work tries to use the sentence or document context for correcting errors and resolving ambiguities, e.g., [Hul92, KEW91, Hon95].

ZD95]. Correction of a given input word is divided in two steps, In a first step, the similarity key or the index is used for coarse search, extracting a list of dictionary words that is guaranteed to contain all interesting corrections of the input string. In the second step (fine search), for each candidate the distance to the garbled input word is computed, using a fine-graded measure. Candidates are ranked according to this distance and the best candidates are suggested as correction words.

Oflazer [Ofl96] suggested another method that can deal even with infinite dictionaries of agglutinating languages. The set of all dictionary words is treated as a regular language over the alphabet of letters. As a prerequisite, a deterministic finite state automaton recognizing this language has to be given.[2] Faced with an input word W, Oflazer starts an exhaustive traversal of the dictionary automaton. At each step, the prefix of all letters that are consumed on the path from the initial state to the current state is maintained. A variant of the Wagner-Fisher algorithm is used to control the walk through the automaton in such a way that only prefixes are generated that potentially lead to a correction candidate V where the Levenshtein-distance between V and W does not exceed a fixed bound n. Each dictionary word V within the given distance to W is added to the output list. Oflazer shows that for bounds $n = 1, 2, 3$ the control mechanism helps to avoid the inspection of most of the states of the dictionary automaton. The method leads to an efficient generation of an appropriate list of correction candidates, even for very large — or infinite — dictionaries.

The first correction procedure that we suggest in this paper can be considered as a variant of Oflazer's approach. We also assume that the dictionary is represented as a deterministic finite state automaton. However, we completely avoid the computation of the Levenshtein-distance during the traversal of the automaton. Given the input word W and a bound k, we first compute a deterministic finite state automaton A that accepts exactly all words V where the Levenshtein-distance between V and W does not exceed k. A is called a Levenshtein-automaton for W. Levenshtein-automaton and dictionary automaton are then traversed in parallel. In this way, each move in the dictionary automaton is controlled by the Levenshtein-automaton and vice versa. We obtain the intersection of the languages of the two automata as our list of correction candidates. Clearly, this intersection is the set of all dictionary words V where the Levenshtein-distance between V and W does not exceed n.

Our main algorithmic result shows that for any fixed degree n and input W a deterministic Levenshtein-automaton A_W for W can be computed in linear time and space in $|W|$. In order to maximize practical efficiency, the computation of A_W for fixed distance bound n is based on a pre-compiled table T_n that contains a parametric and generic description of states and transitions of A_W. At runtime, given input W, parametric states and transitions of T_n are instantiated, yielding the automaton A_W. The instantiation of the parametric transition rules of T_n is triggered by Boolean vectors that characterize the distribution of letters of W in subwords of length $2n+1$. The table-based approach leads to an improved variant of the correction method where the traversal of the dictionary automaton is controlled using the table T_n itself. Moves in A_W are simulated and the actual computation of the Levenshtein-automaton A_W is avoided, thus improving efficiency.

The above results always refer to the "standard" Levenshtein-distance where the distance between two words W and V is defined as the minimal number of insertions,

[2]For finite dictionaries, an efficient algorithm for computing the minimal deterministic finite state automaton for the dictionary has been described in [Mih98, DMWW00].

deletions and substitutions that are needed to transform W into V. For specific applications, variants of this metrics are preferable. In a typesetting context often two symbols are transposed. In the context of OCR-recognition, two symbols are often merged into one symbol, or conversely one symbol is split into two symbols. Motivated by these cases we also study Levenshtein-automata for the modified Levenshtein-distance where insertions, deletions, substitutions and transpositions are used as primitive edit operations, and for the variant where insertions, deletions, substitutions, merges and splits are treated as primitive edit operations. In both cases, techniques and results obtained for the standard Levenshtein-distance can be lifted.

Our evaluation results show that string correction with (simulated) Levenshtein-automata is in fact very fast. For example, using a Bulgarian dictionary with 870,000 entries and (standard) distance bound $n = 1$, the average time to compute and output all lexical Levenshtein-neighbours of a garbled input word on are around 0.4 milliseconds on a Pentium III. Using a german dictionary including composite nouns with 6.058.198 entries the average time was between 1.3 milliseconds (short words) and 2.5 milliseconds. Further results for modified Levenshtein distances, other distance bounds and other lexicon sizes are given below.

The paper is structured as follows. Chapter 2 gives some general technical background. In Chapter 3 we formally define Levensthein-automata and we describe the first string correction method sketched above in more detail. Section 4 gives a generic description of a deterministic Levenshtein-automaton of arbitrary degree n for arbitrary input word W. In Chapter 5 we show how to use this description to derive tables $T_1, T_2, T_3, \ldots$ which contain parametric descriptions of states and transitions of a deterministic Levenshtein-automata of degree $n = 1, 2, 3 \ldots$ for arbitrary input word W. Using these tables it is trivial to generate a deterministic Levenshtein-automaton for input W in time linear in the length of W. Section 6 discusses the second correction method where the actual computation of the Levenshtein-automaton for the input word W is avoided. Chapter 7 describes computation of Levenshtein-automata for the modified distance where transpositions are treated as primitive edit operations. Chapter 8 describes computation of Levenshtein-automata for the metrics where merges and splits are treated as primitive edit operations. In Chapter 8.3 experimental results for string correction and approximate string matching using Levenshtein-automata are added. We finish with a short Conclusion where we mention some side results of our work and comment on related and future work.

Chapter 2

Formal Preliminaries

We assume that the reader is familiar with the basic notions of formal language theory as described, e.g., in [HU79, Koz97]. As usual, *finite state automata* (FSA) are treated as tuples of the form $A = \langle \Sigma, Q, q_0, F, \Delta \rangle$ where Σ is the input alphabet, Q is the set of states, $q_0 \in Q$ is the initial state, F is the set of final states, and $\Delta \subseteq Q \times \Sigma^\varepsilon \times Q$ is the transition relation. Here "ε" denotes the empty word and $\Sigma^\varepsilon := \Sigma \cup \{\varepsilon\}$. We write $\mathcal{L}(A)$ for the language accepted by A.

A finite state automaton A is *deterministic* if the transition relation is a function $\delta : Q \times \Sigma \to Q$. Let $A = \langle \Sigma, Q, q_0, F, \delta \rangle$ be a deterministic FSA, let $\delta^* : Q \times \Sigma^* \to Q$ denote the generalized transition function, which is defined as usual. For $q \in Q$ we write $\mathcal{L}(q) := \{U \in \Sigma^* \mid \delta^*(q, U) \in F\}$ for the language of all words that lead from q to a final state.

The length of a word W is denoted $|W|$. Regular languages over Σ are defined as usual. With $\mathcal{L}_1 \circ \mathcal{L}_2$ we denote the concatenation of the languages $\mathcal{L}_1$ and $\mathcal{L}_2$.

Two words V and W are called *isomorphic* iff V can be obtained from W by a permutation of the alphabet Σ. The notion of isomorphism carries over to automata in the obvious sense.

The Levenshtein-distance between two words

The Levenshtein-distance between two words is based on the notion of a primitive edit operation. In this paper we first consider the standard Levenshtein-distance. Here the primitive operations are the *substitution* of a symbol by another symbol, the *deletion* of a symbol, and the *insertion* of a symbol. Obviously, given two words W and V over the alphabet Σ, it is always possible to rewrite W into V, using primitive edit operations.

Definition 2.0.1 Let V, W be words over the alphabet Σ. The (standard) *Levenshtein-distance* between V and W is the minimal number of edit operations (substitutions, deletions, or insertions) that are needed to transform V into W.

With $d_L(V, W)$ we denote the Levenshtein-distance between V and W. It can be computed using the following simple dynamic programming scheme (cf. [WF74]):

$$d_L(\varepsilon, W) = |W|$$

$$\begin{array}{rcl} d_L(V,\varepsilon) & = & |V| \\ d_L(aV,bW) & = & \begin{cases} d_L(V,W) & \text{if } a=b \\ 1+\min(d_L(V,W),d_L(aV,W),d_L(V,bW)) & \text{if } a\neq b \end{cases} \end{array}$$

for $V,W \in \Sigma^*$ and $a,b \in \Sigma$. The following simple observation follows immediately.

Lemma 2.0.2 *Let $W = UW'$ and $V = UV'$. Then $d_L(V,W) = d_L(V',W')$.*

Let $W = x_1x_2\cdots x_w$ and $V = y_1y_2\cdots y_v$ be two words with Levenshtein-distance $n \geq 0$. Consider a sequence ν of edit operations of minimal length leading from W to V. If we substitute a letter x_i by another symbol z, the latter symbol will not be erased or subsituted by one of the following edit operations of ν since otherwise ν would not have minimal length. Hence there exists a unique letter y_j of V that represents the descendant of z in V and the substitution result of x_i. In the so-called *trace representation* (cf. [WF74]) of ν we introduce a stroke from x_i to y_j. Similarly we introduce a stroke from x_i to y_j if x_i is not touched by any edit operation and if y_j represents the descendant of x_i in the new word V. Assume that all strokes of the above form are introduced. Clearly, two strokes never cross. Moreover, each letter x_i of W that does not represent the starting point of a stroke is deleted by some operation of ν, and each letter y_j of V that does not represent the end point of a stroke is an inserted symbol.

Remark 2.0.3 Let $W = x_1x_2\cdots x_w$ and $V = y_1y_2\cdots y_v$ be two words with Levenshtein-distance $n \geq 1$. Assume that neither V is a prefix of W nor vice versa. Let $U = x_1x_2\cdots x_i$ (where $0 \leq i \leq v,w$) denote the maximal common prefix of V and W. Then, in any trace representation of a minimal sequence ν of edit operations leading from W to V exactly one of the following three cases holds:

1. *Insertion case.* A stroke is starting at x_{i+1} that points to some y_{i+j} where $j > 1$,
2. *Substitution case.* Letters x_{i+1} and y_{i+1} are connected by a stroke,
3. *Deletion case.* A stroke is ending at y_{i+1} that starts at some x_{i+j} where $j > i+1$.

In fact, the only remaining case would be the situation where neither x_{i+1} nor y_{i+1} represent the end point of a stroke. This would mean that x_{i+1} is deleted and y_{i+1} is inserted in ν. Using one substitution instead, we would get a shorter sequence of edit operations from V to W, which gives a contradiction. The three possible cases are indicated in Figure 2.1.

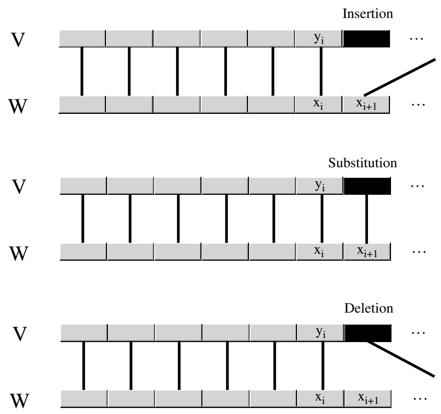


Figure 2.1: Possible trace pictures for situation of Remark 2.0.3.

Chapter 3

String correction with Levenshtein-automata

As indicated in the introduction, we face a situation where we use an eletronic dictionary for detecting and correcting misspelled words. Given any input word W, it is first checked if W is a word of the lexicon. In the negative case, the lexicon is used to generate a list of candidate corrections. The words V of the lexicon that are most similar to W are considered to be good correction candidates. Dissimilarity is measured in terms of the Levenshtein-distance between W and V.

In the sequel, Σ denotes the background alphabet. We assume that the dictionary is implemented in the form of a deterministic FSA or a trie. A trie can be considered as a finite state automaton as well. The language of the automaton represents the set of all correct words. We assume that the automaton has the form $A_D = \langle \Sigma, Q^D, q_0^D, F^D, \delta^D \rangle$. A_D will be called the *dictionary automaton* in the sequel.

Definition 3.0.4 Let W be a word over the alphabet Σ. With $\mathcal{L}_{Lev}(n, W)$ we denote the set of all words $V \in \Sigma^*$ such that $d_L(W, V) \leq n$.

We now introduce the central concept of this paper.

Definition 3.0.5 Let W be a word over the alphabet Σ, let $n \in \mathbb{N}$. A finite state automaton A is a *Levenshtein-automaton of degree n for W* iff $\mathcal{L}(A) = \mathcal{L}_{Lev}(n, W)$.

The first correction method suggested in this paper follows a simple idea. In order to generate a list of correction candidates for a garbled input word W, we select a number n and compute a deterministic Levenshtein-automaton $A_W = \langle \Sigma, Q^W, q_0^W, F^W, \delta^W \rangle$ of degree n for W. Using the following simple backtracking procedure, we traverse the two automata A_W and A_D in parallel.

```
push (<ε, q_0^D, q_0^W>);
while not empty(stack) do begin
   pop (<V, q^D, q^W>);
   for x in Σ do begin
```

```
    q1^D := δ^D(q^D, x);
    q1^W := δ^W(q^W, x);
    if (q1^D <> NIL) and (q1^W <> NIL) then begin
      V1 := concat(V, x);
      push(<V1, q1^D, q1^W>);
      if (q1^D ∈ F^D) and (q1^W ∈ F^W) then output(V1);
    end;
  end;
end;
```

Starting with the pair of initial states $\langle q_0^D, q_0^W \rangle$ and the empty word ε, each step of the traversal adds a new letter $x \in \Sigma$ to the actual word V and leads from a pair of states $\langle q^D, q^W \rangle \in Q^D \times Q^W$ to $\langle \delta^D(q^D, x), \delta^W(q^W, x) \rangle$. We proceed as long as both components are distinct from the empty failure state[1] NIL. Whenever in both automata a final state is reached, the actual word is added to the output.

It is trivial to see that the list of all output words is $\mathcal{L}(A_D) \cap \mathcal{L}(A)$, hence it contains exactly the "grammatical" words in $\mathcal{L}_{Lev}(n, W)$. With a good choice of n, we obtain an appropriate set of correction candidates for the input W.

We shall also introduce a second and related correction method. This method, which avoids the actual computation of Levenshtein-automata, can only be described later, once we have introduced a number of additional concepts.

[1] A *failure state* is a state q whose language $\mathcal{L}(q)$ is empty.

Chapter 4

A family of deterministic Levenshtein-automata

In this chapter we introduce a deterministic Levenshtein-automaton $LEV_n(W)$ of degree n for an input word W. The description is generic in the sense that we neither make any specific assumption on the degree n, nor on the length or the form of the input word W. The description will be the basis for efficient computation of Levenshtein-automata for fixed degree n, to be described in the following section.

Profile sequences and characteristic vectors

We first introduce some notions that help to characterize the structural properties of the input word W that determine the structure of the automaton $LEV_n(W)$.

Definition 4.0.6 Let $U = z_1 \cdots z_u \in \Sigma^u$ be a sequence of characters. The *profil* $Pr(U)$ of U is the sequence of naturals $(n_1 \cdots n_u)$ obtained in the following way. Define $n_1 := 1$. Assume that $n_1, \ldots, n_k$ are defined for some $1 \leq k < u$. If $x_{k+1} \in \{x_1, \ldots, x_k\}$, say, $x_{k+1} = x_i$ (where $1 \leq i \leq k$), then $n_{k+1} := n_i$. In the other case we define $n_{k+1} := \max\{n_i \mid 1 \leq i \leq k\} + 1$.

Example 4.0.7 We have $Pr(aachen) = (112345)$, $Pr(odd) = (122)$, and $Pr(even) = (1213)$.

Let W and W' denote two words of the same length. It should be clear that for any fixed degree n we can use isomorphic deterministic Levenshtein-automata for input words W and W' whenever $Pr(W) = Pr(W')$. A stronger relationship can be established. We shall see that the structure of the deterministic Levenshtein-automaton for an input word W to be described below depends — in a sense to be made precise — only on *local* subprofiles of the input word.

Definition 4.0.8 Let $U = z_1 \cdots z_u$, let $k \geq 1$. The *k-profile sequence* of W is the sequence of profiles

$$Pr(z_1 \cdots z_k) Pr(z_2 \cdots z_{k+1}) \cdots Pr(z_{u-k+1} \cdots z_u)$$

for $k \leq u$. For $k > u$, the k-profile sequence of U is $Pr(z_1 \cdots z_u)$.

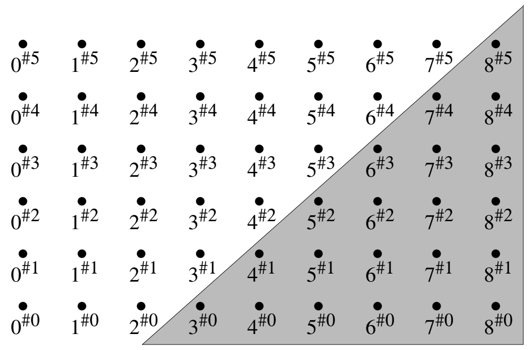


Figure 4.1: Positions and accepting positions.

The k-profile sequences of two words can be identical even for non-isomorphic words.

Example 4.0.9 The 3-profile sequence of *butter* is $(1,2,3),(1,2,2),(1,1,2),(1,2,3)$. The 3-profile sequence of *setter* is the same sequence.

The following notion plays a key role when defining the images of the states of Levenshtein-automata under input symbols $x \in \Sigma$.

Definition 4.0.10 Let $x \in \Sigma$ and let $V = y_1 \ldots y_v \in \Sigma^*$. The *characteristic vector* of x with respect to V is the bit-vector $\chi(x,V) := \langle b_1, \ldots, b_v \rangle$ where $b_j := 1$ iff $y_j = x$ and $b_j := 0$ otherwise.

The following remark shows how the information contained in a profile can be modularized using characteristic vectors. The technique will be used when we define the transitions of $LEV_n(W)$.

Remark 4.0.11 Given the profile of a word V we can derive all characteristic vectors of the form $\chi(x,V)$, just using the characteristic vectors of number $1,2,\ldots$ with respect to $Pr(V)$. For example, if $Pr(V) = (1,2,1,2,3,1,2)$, then the characteristic vectors $\chi(x,V)$ have the form $\langle 1,0,1,0,0,1,0 \rangle$, $\langle 0,1,0,1,0,0,1 \rangle$, $\langle 0,0,0,0,1,0,0 \rangle$ and $\langle 0,0,0,0,0,0,0 \rangle$ (assuming that Σ has at least four letters). Conversely, given the set of all characteristic vectors of the form $\chi(x,V)$ we may obviously derive $Pr(V)$.

Positions and states

We fix an arbitrary input word $W = x_1 \cdots x_w$ and a number $n \in \mathbb{N}$ that denotes the maximal Levenshtein-distance that we want to capture. Numbers $i \in 0,\ldots,w$ will be called the *boundaries* of W. The states of the $\mathcal{L}_{Lev}(n,W)$ are composed of symbolic expressions of a special kind.

Definition 4.0.12 A *position* is an expression of the form $i^{\sharp e}$ where $0 \le i \le w$ and $0 \le e \le n$. Position $i^{\sharp e}$ is *raised* iff $e > 0$, otherwise it is called a *base position.*

Intuitively, an exponent $\sharp e$ is meant to denote a situation where e edit operations have occurred.

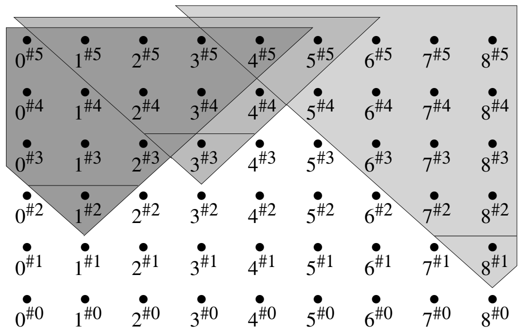


Figure 4.2: Subsumption triangles.

Definition 4.0.13 A position $i^{\sharp e}$ is *accepting* iff $w - i \leq n - e$.

Example 4.0.14 For $n = 5$ and $w = 8$, the set of all positions is depicted in Figure 4.1. Accepting positions are marked.

Definition 4.0.15 A position $i^{\sharp e}$ *subsumes* a position $j^{\sharp f}$ iff $e < f$ and $|j - i| \leq f - e$. The set of all positions that are subsumed by $i^{\sharp e}$ is called the *subsumption triangle* of $i^{\sharp e}$.

Example 4.0.16 Let $n = 5$ and assume that $w = 8$. Figure 4.2 illustrates the subsumption triangles of $1^{\sharp 2}$, $3^{\sharp 3}$ and $8^{\sharp 1}$. Since subsumption is irreflexive, the positions $1^{\sharp 2}$, $3^{\sharp 3}$ and $8^{\sharp 1}$ do not belong to the respective triangles.

The following lemma indicates the background for the notion of subsumption.

Lemma 4.0.17 *Let $W = x_1 \cdots x_w$ and n as above. Let Φ denote the function that assigns to each position $i^{\sharp e}$ the language*

$$\Phi(i^{\sharp e}) := \mathcal{L}_{Lev}(n - e, x_{i+1} \cdots x_w).$$

Let $\pi := i^{\sharp e}$ and $\pi' := j^{\sharp f}$ be two distinct positions. If π subsumes π', then $\Phi(\pi')$ is a subset of $\Phi(\pi)$.

Proof. Assume that $\pi = i^{\sharp e}$ subsumes $\pi' = j^{\sharp f}$. Then $e < f$ and $|j - i| \leq f - e$. Since $x_{j+1} \cdots x_w$ can be obtained from $x_{i+1} \cdots x_w$ by a series of $|j - i|$ insertions (for $j \leq i$) or deletions (for $j > i$) it follows easily that $\Phi(\pi')$ is a subset of $\Phi(\pi)$. □

The states of $\mathcal{L}_{Lev}(n, W)$ are sets of positions of a particular type.

Definition 4.0.18 Let $0 \leq i \leq w$. A *state with base position* $i^{\sharp 0}$ is a set M of positions, not necessarily containing $i^{\sharp 0}$, that safisfies the following properties:

1. for each position $j^{\sharp e}$ in M we have $|i - j| \leq e$. I.e., each position of M, with the possible exception of $i^{\sharp 0}$, lies in the subsumption triangle of $i^{\sharp 0}$.

2. M does not contain any position that is subsumed by another element of M.

Let us note that a state may have several possible base positions.

Example 4.0.19 First assume that $n = 1$ and $w = 2$. Then the states are $\emptyset$, $\{0^{\sharp 0}\}$, $\{1^{\sharp 0}\}$, $\{2^{\sharp 0}\}$, $\{0^{\sharp 1}\}$, $\{1^{\sharp 1}\}$, $\{2^{\sharp 1}\}$, $\{0^{\sharp 1}, 1^{\sharp 1}\}$, $\{0^{\sharp 1}, 2^{\sharp 1}\}$, $\{1^{\sharp 1}, 2^{\sharp 1}\}$, and $\{0^{\sharp 1}, 1^{\sharp 1}, 2^{\sharp 1}\}$. Assume now that $w = 0$. Let n be any natural number. Then the set of non-empty states is $\{\{0^{\sharp e}\} \mid 0 \leq e \leq n\}$. Third, assume that $n = 0$. Let w be any natural number, denoting the length of the input word. Then the set of non-empty states is $\{\{i^{\sharp 0}\} \mid 0 \leq i \leq w\}$.

Definition 4.0.20 Let M be a non-empty state. The minimal number i such that M contains a position of the form i^e (for some e) is called the *minimal boundary* of M.

It is trivial to verify the following lemma.

Lemma 4.0.21 *Let M be a state with minimal boundary i and let $j^{\sharp f} \in M$. Then $j - i \leq n + f$.*

At various places we shall consider the union of two states M and N with a common base position $i^{\sharp 0}$. We write $M \sqcup N$ for the set that is obtained from $M \cup N$ by omission of states that are subsumed by other states. Since the subsumption relation is well-founded, this operation is well-defined. Note that $M \sqcup N$ is again a state with base position $i^{\sharp 0}$. $M \sqcup N$ will be called the *reduced union* of M and N.

Elementary transitions

The transitions of the Levenshtein-automaton of degree n will be defined with the help of transitions that act on single positions. The latter transitions are called *elementary transitions of degree n*. The image of a position under an elementary transition with an input symbol x depends on the distribution of x in a subword of W.

Definition 4.0.22 Let $W = x_1 \cdots x_w$ as above. Let $\pi := i^{\sharp e}$ be a position, and let $k := \min\{n - e + 1, w - i\}$. The *relevant subword of W for position π*, denoted $W_{[\pi]}$, is the subword $x_{i+1} \cdots x_{i+k}$ of W.

Note that the length of $W_{[\pi]}$ cannot exceed $n + 1$.

Example 4.0.23 Let $w = 8$ and $n = 5$. Then the relevant subwords for positions $2^{\sharp 2}$ and $3^{\sharp 0}$ respectively are $x_3x_4x_5x_6$ and $x_4x_5x_6x_7x_8$, as illustrated in Figure 4.3.

Definition 4.0.24 Let W and n as above. An *elementary transition* assigns to each position $\pi = i^{\sharp e}$ and each symbol $x \in \Sigma$ a state $\delta(i^{\sharp e}, x)$. The complete set of elementary transitions is specified in Table 4.1. Notation $\langle 0, b_2, \ldots, b_k \rangle : j$ indicates that j is the minimal index in $\{2, \ldots, k\}$ where $b_j = 1$. This implies that such an index exists.

The following – informal —comments explain the intuition behind these transitions. In Part (I) of the table for $i \leq w - 1$ we distinguish three situations:

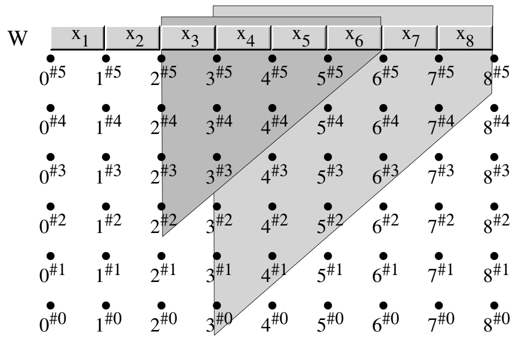


Figure 4.3: Relevant subwords for elementary transitions.

(I) $\quad 0 \leq e \leq n-1$	
$i \leq w-2$	$\delta(i^{\sharp e}, x) := \begin{cases} \{(i+1)^{\sharp e}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1, b_2, \ldots, b_k \rangle, \\ \{i^{\sharp e+1}, (i+1)^{\sharp e+1}, (i+j)^{\sharp e+j-1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0, b_2, \ldots, b_k \rangle : j, \\ \{i^{\sharp e+1}, (i+1)^{\sharp e+1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0, \ldots, 0 \rangle. \end{cases}$
$i = w-1$	$\delta(i^{\sharp e}, x) := \begin{cases} \{(i+1)^{\sharp e}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1 \rangle, \\ \{i^{\sharp e+1}, (i+1)^{\sharp e+1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp e}, x) := \{w^{\sharp e+1}\}$
(II) $\quad e = n$	
$i \leq w-1$	$\delta(i^{\sharp n}, x) := \begin{cases} \{(i+1)^{\sharp n}\} \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1 \rangle, \\ \emptyset \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp n}, x) := \emptyset.$

Table 4.1: Table of elementary transitions for $\pi = i^{\sharp e}$.

1. The first entry of $\chi(x, W_{[\pi]})$ is 1,
2. The first entry of $\chi(x, W_{[\pi]})$ is 0, but $\chi(x, W_{[\pi]})$ has an entry 1, the minimal one has index index j,
3. all entries of $\chi(x, W_{[\pi]})$ are 0.

In Situation 3, x does not occur in $W_{[\pi]}$. The transition can be interpreted as a default transition. Image element $i^{\sharp e+1}$ captures the insertion of x at boundary i, image element $(i+1)^{\sharp e+1}$ captures the substitution of x_{i+1} with x. Other possible explanations for the occurrence of x are covered via subsumption. For example, assume that x_{i+1} is deleted and x_{i+2} is substituted by x. The position reached in this case is $(i+2)^{\sharp e+2}$. We do not add this position to the image set since it is subsumed by $(i+1)^{\sharp e+1}$. Note that default transitions for $e = n$ lead to the failure state $\emptyset$ (cf. Part (II)).

In Situation 2, the image element $i^{\sharp e+1}$ again covers the situation where symbol x is inserted before x_{i+1}. Element $(i+1)^{\sharp e+1}$ covers the situation where x_{i+1} is substituted by x. Element $(i+j)^{\sharp e+j-1}$ covers the situation where the elements

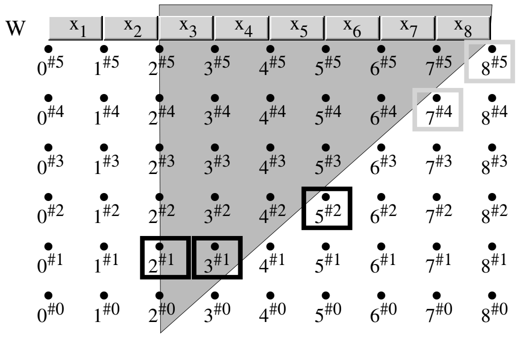


Figure 4.4: Geometric interpretation of elementary transitions.

$x_{i+1}, \ldots, x_{i+j-1}$ are deleted. The reader might wonder why only the entry 1 with minimal index j is essential. This entry corresponds to the first occurrence of x in $W_{[\pi]}$. Assume that $j = j_1 < \ldots < j_h$ is the list of all indices where $\chi(x, W_{[\pi]})$ has an entry 1. In this situation position $(i+j)^{\sharp e+j-1}$ subsumes all positions $(i+j_l)^{\sharp e+j_l-1}$ for $1 \neq l$. Hence, elimination of subsumed positions leads to the state $\{i^{\sharp e+1}, (i+1)^{\sharp e+1}, (i+j)^{\sharp e+j-1}\}$ which is used as image above. See Example 4.0.25 below for an illustration.

In Situation 1 we might expect that the image is rather the set $\{i^{\sharp e+1}, (i+1)^{\sharp e+1}, (i+1)^{\sharp e}\}$. But note that $(i+1)^{\sharp e}$ subsumes both other elements. Hence, via elimination of subsumed positions we arrive at $\{(i+1)^{\sharp e}\}$.

Example 4.0.25 Let $w = 8$ and $n = 5$. In Figure 4.4 we consider the image of $\pi = 2^{\sharp 0}$ under $x \in \Sigma$. We have $W_{[\pi]} = x_3 \cdots x_8$. We assume that $\chi(x, W_{[\pi]}) = \langle 0, 0, 1, 0, 1, 1 \rangle$. This means that x_5, x_7 and x_8 are the symbols of $W_{[\pi]}$ that are identical to x. In this situation we have $\delta(2^{\sharp 0}, x) = \{2^{\sharp 1}, 3^{\sharp 1}, 5^{\sharp 2}\}$ as illustrated in Figure 4.4.

The proof of the following lemma is straightforward.

Lemma 4.0.26 *Let M be a set of positions. Assume that all elements of M are subsumed by $i^{\sharp e}$ where $i < w$. Then the image of any element of M under any elementary transition contains only positions that are subsumed by $(i+1)^{\sharp e}$. If all elements of M are subsumed by $w^{\sharp e}$, then the image of any element of M under any elementary transition has only positions that are subsumed by $w^{\sharp e}$.*

We now ask how the elementary transitions for Levenshtein-automata of different degrees n for the same input word W are related. In order to avoid notational ambiguities we write $\delta^{(n)}$ for the function that describes the elementary transitions for the Levenshtein-automaton of degree n. If e is a natural number we write $[i^{\sharp f}]^{\sharp e}$ for the "lifted" position $i^{\sharp f+e}$ (it will be guaranteed that each such expression in fact denotes a position). If M is a state we define the lifted version $[M]^{\sharp e} := \{[\pi]^{\sharp e} \mid \pi \in M\}$. The following lemma shows that once the elementary transitions from positions $i^{\sharp 0}$ for degrees 0, …, $n-1$ are fixed, the elementary transitions of degree n for positions $i^{\sharp e}$ for $1 \leq e \leq n$ are simply defined by "raising".

Lemma 4.0.27 (Raising Lemma for elementary transitions) *Let $n > 0$ and*

$1 \leq e \leq n$. Then for any position $i^{\sharp e}$ of degree n and any $x \in \Sigma$ we have

$$\delta^{(n)}(i^{\sharp e}, x) = [\delta^{(n-e)}(i^{\sharp 0}, x)]^{\sharp e}.$$

Proof. Since $\min\{n-e+1, w-i\} = \min\{(n-e)-0+1, w-i\}$ it follows that the relevant subword for $i^{\sharp e}$ for degree n is identical to the relevant subword of $i^{\sharp 0}$ for degree $n-e$. We may denote it in the form W_r. First assume that $i \leq w-2$ and the first entry of $\chi(x, W_r)$ is 1. Then we have

$$\begin{array}{rcl}\delta^{(n)}(i^{\sharp e}, x) & = & \{(i+1)^{\sharp e}\} \\ & = & [\{(i+1)^{\sharp 0}\}]^{\sharp e} \\ & = & [\delta^{(n-e)}(i^{\sharp 0}, x)]^{\sharp e}.\end{array}$$

The remaining cases are similar. □

The Levenshtein automaton

We can now introduce the family of Levenshtein-automata that we use in for string correction.

Definition 4.0.28 Let $W = x_1 \cdots x_w$ where $w \geq 0$, let $n \geq 0$. Then $LEV_n(W)$ is the deterministic finite state automaton $\langle \Sigma, Q, q_0, F, \Delta \rangle$ where

1. the set of *states* Q contains all states in the sense of Definition 4.0.18,
2. the *initial state* is $q_0 := \{0^{\sharp 0}\}$,
3. the set F of *final states* contains all states $M \in Q$ that contain an accepting position,
4. the *transition function* Δ is defined in the following way: for any symbol $y \in \Sigma$ and any state $M \in Q$, $\Delta(M, y) := \bigsqcup_{\pi \in M} \delta(\pi, y)$.

It follows from Lemma 4.0.26 that Δ assigns to each state $M \in Q$ and each $y \in \Sigma$ again a state $M \in Q$. In fact, a state with base position $i^{\sharp 0}$ ($i < w$) is always mapped to a state with base position $(i+1)^{\sharp 0}$, and states with base position $w^{\sharp 0}$ are mapped to states with base position $w^{\sharp 0}$. This shows that $LEV_n(W)$ is a deterministic finite state automaton.

As a first step we discuss how the transition functions for Levensthein-automata of different degrees for the same input word W are related. We write $\Delta^{(n)}$ for the transition function of the Levenshtein-automaton of degree n.

Lemma 4.0.29 (Raising Lemma for transitions) *Let $n > 0$ and $1 \leq e \leq n$. Then for state of degree n of the form $[M]^{\sharp e}$ and any $x \in \Sigma$ we have*

$$\Delta^{(n)}([M]^{\sharp e}, x) = [\Delta^{(n-e)}(M, x)]^{\sharp e}.$$

Proof. Using our earlier notation and Lemma 4.0.27 we obtain

$$\Delta^{(n)}([M]^{\sharp e}, x) \quad = \quad \bigsqcup_{\pi \in M} \delta^{(n)}([\pi]^{\sharp e}, x)$$

$$
\begin{aligned}
&= \bigsqcup_{\pi \in M} [\delta^{(n-e)}(\pi, x)]^{\sharp e} \\
&= [\bigsqcup_{\pi \in M} \delta^{(n-e)}(\pi, x)]^{\sharp e} \\
&= [\Delta^{(n-e)}(M, x)]^{\sharp e}
\end{aligned}
$$

The result follows. □

In the sequel, let $LEV_n(W) = \langle \Sigma, Q, q_0, F, \Delta \rangle$ as in Definition 4.0.28.

Proposition 4.0.30 *The following properties hold:*

1. $\mathcal{L}(\emptyset) = \emptyset$,
2. *for all states* M, N *with a common base position and all* $y \in \Sigma$*:*
$$\Delta(M \sqcup N, y) = \Delta(M, y) \sqcup \Delta(N, y),$$
3. *for all states* M, N *with a common base position and all* $V \in \Sigma^*$*:*
$$\Delta^*(M \sqcup N, V) = \Delta^*(M, V) \sqcup \Delta^*(N, V),$$
4. *for all states* $M \subseteq Q \setminus \{\{0^{\sharp 0}\}, \ldots, \{w^{\sharp 0}\}\}$*:* $\mathcal{L}(M) = \bigcup_{\pi \in M} \mathcal{L}(\{\pi\})$.

Proof. Part 1 is trivial.

Proof of Part 2. Since M and N have a common base position it follows that $M \sqcup N$ is again a state. We have

$$
\begin{aligned}
\Delta(M \sqcup N, y) &= \bigsqcup_{\pi \in M \sqcup N} \delta(\pi, y) \\
&= \bigsqcup_{\pi \in M} \delta(\pi, y) \sqcup \bigsqcup_{\pi \in N} \delta(\pi, y) \\
&= \Delta(M, y) \sqcup \Delta(N, y).
\end{aligned}
$$

Proof of Part 3. Follows from Part 2 by a trivial induction on the length of V.

Proof of Part 4. Let RA denote the set of all raised accepting positions. Using Part 3 we obtain

$$
\begin{aligned}
V \in \mathcal{L}(M) &\Leftrightarrow \Delta^*(M, V) \in F \\
&\Leftrightarrow \bigsqcup_{\pi \in M} \Delta^*(\{\pi\}, V) \in F \\
&\Leftrightarrow \exists f \in RA\colon f \in \bigsqcup_{\pi \in M} \Delta^*(\{\pi\}, V) \\
&\overset{*}{\Leftrightarrow} \exists f \in RA\colon f \in \bigcup_{\pi \in M} \Delta^*(\{\pi\}, V) \\
&\Leftrightarrow \exists f \in RA, \exists \pi \in M\colon f \in \Delta^*(\{\pi\}, V) \\
&\Leftrightarrow \exists \pi \in M\colon \Delta^*(\{\pi\}, V) \in F \\
&\Leftrightarrow \exists \pi \in M\colon V \in \mathcal{L}(\{\pi\}) \\
&\Leftrightarrow V \in \bigcup_{\pi \in M} \mathcal{L}(\{\pi\}).
\end{aligned}
$$

To see the marked equivalence notice that all positions in $\bigcup_{\pi\in M}\Delta^*(\{\pi\},V)$ are raised. Each position of this set that subsumes a raised accepting position is itself a raised accepting position. $\square$

Proposition 4.0.31 *For all $0\le i\le w$ and all $0\le e\le n$ we have*

$$\mathcal{L}(\{i^{\sharp e}\})=\mathcal{L}_{Lev}(n-e,x_{i+1}\cdots x_w).$$

Proof. We proceed by induction on n. The case $n=0$ is simple: the set of positions is $\{i^{\sharp 0}\mid 0\le i\le w\}$. Only $w^{\sharp 0}$ is an accepting position. States have the form $\emptyset$ or $\{i^{\sharp 0}\}$ $(0\le i\le w)$. Transitions from states $\{i^{\sharp 0}\}$ can be considered as elementary transitions from positions $i^{\sharp 0}$. By Part 1 of Proposition 4.0.30, $\mathcal{L}(\emptyset)=\emptyset$. Part (II) of Table 4.1 shows that only transitions leading from states $\{i^{\sharp 0}\}$ for $i<w$ with input x_{i+1} to $\{(i+1)^{\sharp 0}\}$ are relevant — all other transitions lead to the failure state $\emptyset$. Hence $\mathcal{L}(\{i^{\sharp 0}\})=\{x_{i+1}\cdots x_w\}=\mathcal{L}_{Lev}(0,x_{i+1}\cdots x_w)$ for all $0\le i\le w$.

Now let $n\ge 1$ and assume that the Proposition is correct for all $0\le n'<n$. The case $e=n$ is simple since all relevant transitions are of the form $\{i^{\sharp n}\}\mapsto\{(i+1)^{\sharp n}\}$ under x_{i+1} $(i<w)$. Hence assume that $e<n$. Since for $1\le e<n$ transitions from states $\{i^{\sharp e}\}$ are defined by raising of transitions of degree $n'=n-e$ (cf. Lemma 4.0.29) the induction hypothesis shows that

$$\mathcal{L}(\{i^{\sharp e}\})=\mathcal{L}_{Lev}(n-e,x_{i+1}\cdots x_w)\quad(0\le i\le w,1\le e\le n).\qquad(\dagger)$$

Hence it only remains to prove that for all $0\le i\le w$ we have

$$\mathcal{L}(\{i^{\sharp 0}\})=\mathcal{L}_{Lev}(n,x_{i+1}\cdots x_w).$$

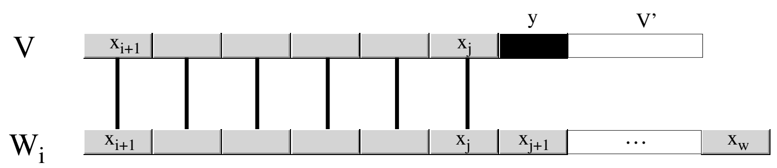
In the sequel, let W_i denote the suffix $x_{i+1}\cdots x_w$ of W $(0\le i\le w)$. From $(\dagger)$ and Lemma 4.0.17 we obtain the following : for all positions $i^{\sharp e}$ and $j^{\sharp f}$ such that $e\ne 0\ne f$: if $i^{\sharp e}$ is subsumed by $j^{\sharp f}$, then $\mathcal{L}(\{i^{\sharp e}\})$ is a proper subset of $\mathcal{L}(\{j^{\sharp f}\})$ $(\dagger\dagger)$.

I. We first show that $\mathcal{L}_{Lev}(n,W_i)\subseteq\mathcal{L}(\{i^{\sharp 0}\})$. Let $V\in\mathcal{L}_{Lev}(n,W_i)$.

Case 1.1: V is obtained from W_i by deleting a suffix of length k $(0\le k\le n)$ of W_i. Starting from state $\{i^{\sharp 0}\}$ and consuming V we reach state $\{(w-k)^{\sharp 0}\}$. Since $(w-k)^{\sharp 0}$ is an accepting position, state $\{(w-k)^{\sharp 0}\}$ is final, hence $V\in\mathcal{L}(\{i^{\sharp 0}\})$.

Case 1.2: W_i is obtained from V by deleting a suffix of length k $(1\le k\le n)$ of V. Starting from state $\{i^{\sharp 0}\}$ and first consuming W_i we reach $\{w^{\sharp 0}\}$. The k additional transitions lead to $\{w^{\sharp k}\}$. Since $w^{\sharp k}$ is an accepting position, the latter state is final, hence $V\in\mathcal{L}(\{i^{\sharp 0}\})$.

Case 2: In the remaining cases there exists an index $j<w$ such that $V=x_{i+1}\cdots x_jyV'$ where $y\ne x_{j+1}$.

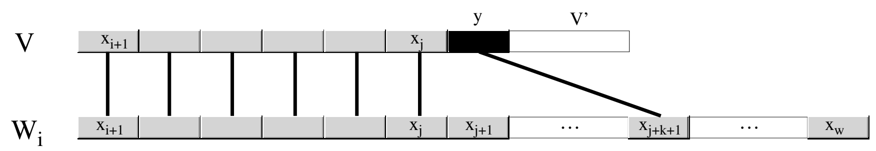


We have $yV'\in\mathcal{L}_{Lev}(n,x_{j+1}\cdots x_w)$ by Lemma 2.0.2. We fix a sequence ν of edit operations leading from $x_{j+1}\cdots x_w$ to yV' of minimal length and consider the three cases described in Remark 2.0.3.

2.1. If the occurrence of y is an insertion before x_{j+1} (where $i \leq j \leq w$), then $V' \in \mathcal{L}_{Lev}(n-1, x_{j+1}\cdots x_w)$. Since $y \neq x_{j+1}$, starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1},\ldots x_j, y$ we reach states $\{(i+1)^{\sharp 0}\},\ldots,\{j^{\sharp 0}\}, M$ where M contains $j^{\sharp 1}$ (cf. elementary transitions). It follows from (†) and Part 4 of Proposition 4.0.30 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.2. If the occurrence of y substitutes x_{j+1} (where $i \leq j < w$), then V' belongs to $\mathcal{L}_{Lev}(n-1, x_{j+2}\cdots x_w)$. Starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1},\ldots x_j, y$ we reach states $\{(i+1)^{\sharp 0}\},\ldots,\{j^{\sharp 0}\}, M$ where M contains $(j+1)^{\sharp 1}$. It follows from (†) and Part 4 of Proposition 4.0.30 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.3. In the remaining cases, by Remark 2.0.3 there exists some $1 \leq k \leq n$ such that we have a stroke from x_{j+k+1} to y in the trace representation of ν.



This means that the k letters $x_{j+1}, x_{j+2},\ldots,x_{j+k}$ are erased. The distance between $x_{j+k+1}\cdots x_w$ and yV' is bounded by $n-k$. In the sequel, let k_0 be the smallest index in $\{1,\ldots,k+1\}$ such that $x_{j+k+1} = x_{j+k_0}$. It follows from the definition of elementary transitions (cf. Table 4.1) that we reach state $M := \{j^{\sharp 1}, (j+1)^{\sharp 1}, (j+k_0)^{\sharp k_0-1}\}$ from $\{j^{\sharp 0}\}$ by consuming $x_{j+k_0} = x_{j+k+1}$.

2.3.1. Assume first that $x_{j+k+1} = y$. Then the distance between $x_{j+k+2}\cdots x_w$ and V' is bounded by $n-k$. By (†), $V' \in \mathcal{L}(\{(j+k+1)^{\sharp k}\})$. Since $(j+k+1)^{\sharp k}$ is subsumed by $(j+k_0)^{\sharp k_0-1}$ also $V' \in \mathcal{L}(\{(j+k_0)^{\sharp k_0-1}\})$, by (††) and $V' \in \mathcal{L}(M)$ by Part 4 of Proposition 4.0.30. It follows that $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.3.2. Assume that $x_{j+k+1} \neq y$. Then the distance between $x_{j+k+2}\cdots x_w$ and V' is bounded by $n-k-1$. Starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1},\ldots x_j, y$ we eventually reach a state M containing $(j+1)^{\sharp 1}$. This position subsumes $(j+k+1)^{\sharp k+1}$. It follows from (†), (††) and Part 4 of Proposition 4.0.30 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

II. It remains to prove that $\mathcal{L}(\{i^{\sharp 0}\}) \subseteq \mathcal{L}_{Lev}(n, W_i)$ for $0 \leq i \leq w$. Let $V \in \mathcal{L}(\{i^{\sharp 0}\})$. If V is accepted - starting from $\{i^{\sharp 0}\}$ - on a path of singleton sets with basic positions $\{(i+1)^{\sharp 0}\}, \{(i+2)^{\sharp 0}\},\ldots,\{k^{\sharp 0}\}$, then $k^{\sharp 0}$ is accepting which implies that V has the form $x_{i+1}\cdots x_k$ where $w-k \leq n$. This shows that $V \in \mathcal{L}_{Lev}(n, W_i)$. In the other case, starting from state $\{i^{\sharp 0}\}$ and consuming the prefix $V'y$ of $V = V'yV''$ we reach states $\{(i+1)^{\sharp 0}\},\ldots\{j^{\sharp 0}\}, M$ where $M \neq \{(j+1)^{\sharp 0}\}$.

Case (a): $j < w$ and M has the form $\{j^{\sharp 1}, (j+1)^{\sharp 1}\}$. In this case, by (†) and Part 4 of Proposition 4.0.30, either V'' has distance $\leq n-1$ to $x_{j+1}\cdots x_w$ or V'' has distance $\leq n-1$ to $x_{j+2}\cdots x_w$. In the former case, with an additional insertion of y we see that V has distance $\leq n$ to W_i. In the latter case, using a substitution $x_{j+1} \mapsto y$ we see V has distance $\leq n$ to W_i.

Case (b): $j < w$ and M has the form $\{j^{\sharp 1}, (j+1)^{\sharp 1}, (j+k)^{\sharp k-1}\}$. Here we have to consider the additional case where V'' has distance $\leq n-(k-1)$ to $x_{j+k+1}\cdots x_w$. However, we know that $y = x_{j+k}$. Deleting $x_{j+1},\ldots,x_{j+k-1}$ we see that yV' has distance $\leq n$ to $x_{j+1}\ldots x_w$, hence the same holds for V and W_i.

Case (c): $j = w$. In this case $M = \{w^{\sharp 1}\}$ and $V' = W_i$. It follows from (†) that V'' has distance $\leq n-1$ to the empty word ε. Hence V has distance $\leq n$ to W_i. □

Theorem 4.0.32 *$LEV_n(W)$ is a deterministic and acyclic Levenshtein-automaton of degree n for W. For fixed degree n, the size of $LEV_n(W)$ is linear in $|W|$.*

Proof. Proposition 4.0.31 shows that

$$\mathcal{L}(LEV_n(W)) = \mathcal{L}(\{0\}) = \mathcal{L}_{Lev}(n, W),$$

hence $LEV_n(W)$ is a deterministic Levenshtein-automaton of degree n for W. If $j^{\sharp f}$ is in the image set of position $i^{\sharp e}$, then $i + e < j + f$. Hence it is easy to see that $LEV_n(W)$ is acyclic. Obviously the number of possible base positions for states is linear in $|W|$, and for fixed degree n there exists a uniform bound on the number of distinct states with a fixed base position $i^{\sharp 0}$. It follows that the number of states of $LEV_n(W)$ is linear in $|W|$. Since the alphabet Σ is fixed, the size of $LEV_n(W)$ is linear in $|W|$. □

The rest of this chapter will be used to introduce some notions that help to obtain a concrete description of the transition function Δ of $LEV_n(W)$ in the situation where the degree n is fixed. Recall that the above definition of Δ is indirect in the sense that the image of a state is only defined in terms of the images of its members under elementary transitions. Clearly, if M is a state and $x \in \Sigma$, in order to directly define the image $\Delta(M, x)$ we have to distinguish appropriate subcases that take the distribution of the occurrences of x in W into account. As it turns out, it suffices to consider the occurrences of x in a particular subword of W.

Definition 4.0.33 Let W and n as above. Let M be a non-empty state with minimal boundary i. Let $k := \min\{2n+1, w-i\}$. The *relevant subword of M*, denoted $W_{[M]}$, is the subword $x_{i+1} \cdots x_{i+k}$ of W.

Since the relevant subword does not depend on the state M itself, but only on the minimal boundary i, we also write $W_{[i]}$ for $W_{[M]}$. Note that the length of $W_{[i]}$ cannot exceed $2n+1$. It follows from Lemma 4.0.21 and from Definitions 4.0.22 and 4.0.33 that for each position $\pi \in M$ always $W_{[\pi]}$ is a subword of $W_{[M]}$. Table 4.1 shows that for any position π the image $\delta(\pi, x)$ only depends on $\chi(x, W_{[\pi]})$. Thus, given a state M, the image $\Delta(M, x)$ is completely determined by the characteristic vector $\chi(x, W_{[M]})$. In the following chapter we shall see that for fixed degree n this observation can be used to describe Δ in terms of a finite table.

Remark 4.0.34 The transition function Δ is completely determined by the characteristic vectors $\chi(x, W_{[i]})$ of symbols $x \in \Sigma$ with respect to the subwords $W_{[i]}$ of the form $W_{[i]} = x_{i+1} \cdots x_{i+k}$ where $k := \min\{2n+1, w-i\}$. If W and W' are two words of the same length, and if the $(2n+1)$-profile sequences of W and W' are identical, then $LEV_n(W)$ and $LEV_n(W')$ are isomorphic modulo transition labels.

Chapter 5

Computation of deterministic Levenshtein-automata of fixed degree

The general description of the Levensthein-automaton $LEV_n(W)$ given in the previous chapter can be used to derive, for any *fixed* bound n, an algorithm that actually computes the automaton $LEV_n(W)$ in linear time, given any input word W. The principle will first be illustrated for degree $n = 1$.

5.1 Computing the Levensthein-automaton of degree 1

Using the general description of $LEV_n(W)$ we derive a generic description of $LEV_1(W)$ for *arbitrary* input W in terms of

- a parametric list of states, with a fixed initial state $\{0^{\sharp 0}\}$,
- a parametric list of final states,
- a table T_1 which gives a parametric description of the transition function Δ.

Parametric list of states and final states

For input $W = x_1 \cdots x_w$ and $n = 1$ the list of positions is

$$0^{\sharp 0}, \ldots, w^{\sharp 0},$$
$$0^{\sharp 1}, \ldots, w^{\sharp 1}.$$

It follows easily from Definition 4.0.18 that we have the following states:

$$\begin{array}{lll} \emptyset & & \text{failure state}, \\ A_i & := & \{i^{\sharp 0}\} \quad (0 \leq i \leq w), \\ B_i & := & \{i^{\sharp 1}\} \quad (0 \leq i \leq w), \end{array}$$

$$
\begin{array}{rcll}
C_i & := & \{i^{\sharp 1}, (i+1)^{\sharp 1}\} & (0 \leq i \leq w-1), \\
D_i & := & \{i^{\sharp 1}, (i+2)^{\sharp 1}\} & (0 \leq i \leq w-2), \\
E_i & := & \{i^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} & (0 \leq i \leq w-2).
\end{array}
$$

The initial state is A_0. Accepting positions are $w^{\sharp 1}$, $w^{\sharp 0}$, as well as $(w-1)^{\sharp 0}$ for $w \geq 1$. It follows immediately that the final states are

$$
\begin{array}{rl}
A_w, A_{w-1}, B_w, C_{w-1}, D_{w-2}, E_{w-2} & \text{for } w \geq 2, \\
A_w, A_{w-1}, B_w, C_{w-1} & \text{for } w = 1, \\
A_w, B_w & \text{for } w = 0.
\end{array}
$$

Parametric description of the transitions function

In order to derive the parametric description of the transition function we first refine the general description of elementary descriptions given in Table 4.1. For the case $n = 1$, we obtain the set of elementary transitions given in Table 5.1. Using this

(I)	$e = 0$
$i \leq w-2$	$\delta(i^{\sharp 0}, x) := \begin{cases} \{(i+1)^{\sharp 0}\} \\ \quad \text{for } \chi(x, x_{i+1}x_{i+2}) = \langle 1, b_2 \rangle, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \\ \quad \text{for } \chi(x, x_{i+1}x_{i+2}) = \langle 0, 1 \rangle, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}\} \\ \quad \text{for } \chi(x, x_{i+1}x_{i+2}) = \langle 0, 0 \rangle. \end{cases}$
$i = w-1$	$\delta(i^{\sharp 0}, x) := \begin{cases} \{(i+1)^{\sharp 0}\} \\ \quad \text{for } \chi(x, x_{i+1}) = \langle 1 \rangle, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}\} \\ \quad \text{for } \chi(x, x_{i+1}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp 0}, x) := \{w^{\sharp 1}\}$
(II)	$e = 1$
$i \leq w-1$	$\delta(i^{\sharp 1}, x) := \begin{cases} \{(i+1)^{\sharp 1}\} \quad \text{for } \chi(x, x_{i+1}) = \langle 1 \rangle, \\ \emptyset \quad \text{for } \chi(x, x_{i+1}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp 1}, x) := \emptyset.$

Table 5.1: Table of elementary transitions for degree 1.

table it is simple to compute a parametric description of the full transition function Δ, following the remarks at the end of the previous chapter. The description is given in Table 5.2. Images $\Delta(M, x)$ are specified using a subcase analysis where the possible characteristic vectors $\chi(x, W_{[M]})$ are distinguished. The following example shows how the entries of Table 5.2 are computed.

Example 5.1.1 Let us ask for the image $\Delta(C_i, x)$ of state $C_i = \{i^{\sharp 1}, (i+1)^{\sharp 1}\}$ under $x \in \Sigma$, assuming that that $i \leq w-3$ and $\chi(x, W_{[C_i]}) = \chi(x, x_{i+1}x_{i+2}x_{i+3}) = \langle 1, 1, 0 \rangle$. We have $W_{[i^{\sharp 1}]} = x_{i+1}$, $W_{[(i+1)^{\sharp 1}]} = x_{i+2}$, hence $\chi(x, W_{[i^{\sharp 1}]}) = \langle 1 \rangle = \chi(x, W_{[(i+1)^{\sharp 1}]})$. Using Table 5.1 we obtain

$$
\begin{array}{rcl}
\Delta(C_i, x) & = & \delta(i^{\sharp 1}, x) \sqcup \delta((i+1)^{\sharp 1}, x) \\
& = & \{(i+1)^{\sharp 1}\} \sqcup \{(i+2)^{\sharp 1}\} \\
& = & \{(i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \\
& = & C_{i+1}.
\end{array}
$$

$0 \leq i \leq w-3$					
$\chi(x, x_{i+1}x_{i+2}x_{i+3})$	A_i	B_i	C_i	D_i	E_i
$\langle 0,0,0\rangle$	C_i	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\langle 1,0,0\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}
$\langle 0,1,0\rangle$	E_i	$\emptyset$	B_{i+2}	$\emptyset$	B_{i+2}
$\langle 0,0,1\rangle$	C_i	$\emptyset$	$\emptyset$	B_{i+3}	B_{i+3}
$\langle 1,1,0\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	B_{i+1}	C_{i+1}
$\langle 1,0,1\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	D_{i+1}	D_{i+1}
$\langle 0,1,1\rangle$	E_i	$\emptyset$	B_{i+2}	B_{i+3}	C_{i+2}
$\langle 1,1,1\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	D_{i+1}	E_{i+1}

$i = w-2$					
$\chi(x, x_{i+1}x_{i+2})$	A_i	B_i	C_i	D_i	E_i
$\langle 0,0\rangle$	C_i	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\langle 1,0\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}
$\langle 0,1\rangle$	E_i	$\emptyset$	B_{i+2}	$\emptyset$	B_{i+2}
$\langle 1,1\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	B_{i+1}	C_{i+1}

$i = w-1$			
$\chi(x, x_{i+1})$	A_i	B_i	C_i
$\langle 0\rangle$	C_i	$\emptyset$	$\emptyset$
$\langle 1\rangle$	A_{i+1}	B_{i+1}	B_{i+1}

$i = w$		
$\chi(x, \varepsilon)$	A_i	B_i
$\langle\rangle$	B_i	$\emptyset$

Table 5.2: Table T_1: parametric transitions for $LEV_1(W)$.

In the same way, all other entries of Table 5.2 can be computed.

Computation of the actual automaton

Obviously, given the above generic description of LEV_1 it is possible to generate for any concrete input W the automaton $LEV_1(W)$ in time $O(|W|)$.

Theorem 5.1.2 *There exists an algorithm that computes for any input word W the automaton $LEV_1(W)$ in time and space $O(|W|)$.*

Corollary 5.1.3 *For any input W, the minimal deterministic Levenshtein-automaton of degree 1 for W can be computed in time and space $O(|W|)$.*

Proof. A result by D. Revuz [Rev92] shows that acyclic deterministic finite state automata can be minimalized in linear time. Since $LEV_1(W)$ is deterministic and acyclic the result follows. □

Example 5.1.4 Figure 5.1 describes the automaton $LEV_1(W)$ for the input word "atlas". For each word W of length 5 with 3-profile sequence $(1,2,3),(1,2,3),(1,2,3)$ the automaton $LEV_1(W)$ has the same structure, modulo renaming of transition labels. Similarly Figure 5.2 describes the structure of $LEV_1(W)$ for the word "otter".

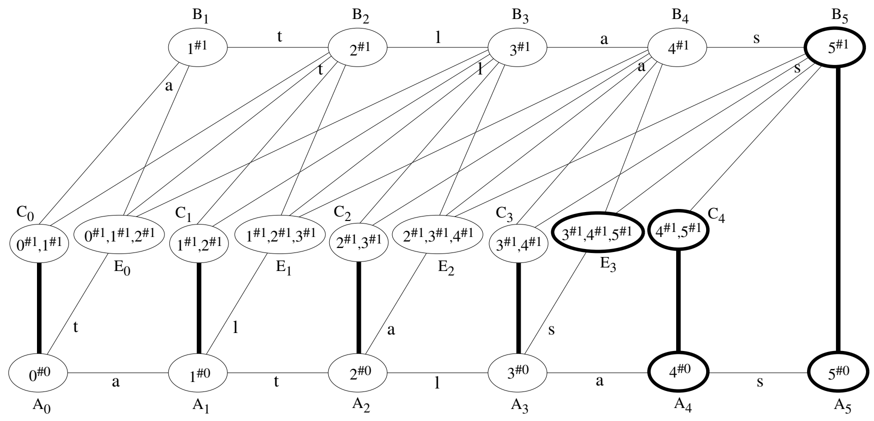


Figure 5.1: Deterministic Levenshtein-automaton $LEV_1(W)$ for input $W =$ "atlas".

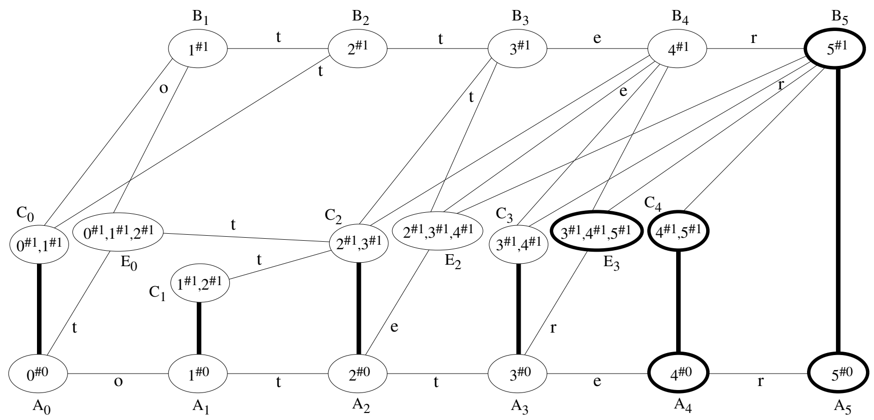


Figure 5.2: Deterministic Levenshtein-automaton $LEV_1(W)$ for input $W =$ "otter".

Here for each word W of length 5 with 3-profile sequence $(1, 2, 2), (1, 1, 2), (1, 2, 3)$ the automaton $LEV_1(W)$ has the same structure, modulo renaming of transition labels.

5.2 Computing Levensthein-automata of higher degree

For any fixed degree $n \geq 2$, the computation of $LEV_n(W)$ essentially follows the same ideas as in the case $n = 1$. Given the degree n, an *offline-computation* is used to compute

1. a parametric description of the set of all states of $LEV_n(W)$ for arbitrary input word W, using the minimal boundary i of states as a parameter,

2. a parametric description of the set of all finite states,

3. a parametric transition table T_n that define the images of parametric states M under input $x \in \Sigma$, subject to the form of the characteristic $\chi(x, W_{[M]})$.

In each case, the initial state is $\{0^{\sharp 0}\}$. Once we have the parametric description of $LEV_n(W)$ for arbitrary W at our disposal, we may use it to compute for any concrete input word W the automaton $LEV_n(W)$ in time linear in $|W|$.

Theorem 5.2.1 *For any fixed degree n, there exists an algorithm that computes for input word W the automaton $LEV_n(W)$ in time and space $O(|W|)$.*

Corollary 5.2.2 *For any input W, the minimal deterministic Levenshtein-automaton of fixed degree n for W can be computed in time and space $O(|W|)$.*

Proof. As in the case $n = 1$. □

Remark 5.2.3 Whereas five parametric states (i.e., A_i, B_i, B_i, D_i, and E_i) are sufficient for degree $n = 1$, the number of parametric states that are needed for degrees $2, 3, 4, \ldots$ grows quickly. For $n = 2$ there are 30 parametric states (ignoring state $\emptyset$), which are listed in Example 5.2.4. Since relevant subwords $W_{[M]}$ may have length $2n + 1 = 5$ the boolean vectors that have to be considered when defining the transition function have maximal length 5. Hence the maximal subtable of Δ has dimension 30×32. For $n = 3$, the number of parametric states is 196, the maximal subtable for Δ has dimension 196×128. For $n = 4$, there are alread 1353 parametric states, the maximal subtable for Δ has dimension 1352×512.

Example 5.2.4 The non-empty states of $LEV_2(W)$ are the following:

$$
\begin{array}{rl}
\{i^{\sharp 0}\} & (0 \le i \le w), \\
\{i^{\sharp 1}\} & (0 \le i \le w), \\
\{i^{\sharp 1}, (i+1)^{\sharp 1}\} & (0 \le i \le w-1), \\
\{i^{\sharp 1}, (i+2)^{\sharp 1}\} & (0 \le i \le w-2), \\
\{i^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} & (0 \le i \le w-2), \\
\{i^{\sharp 2}, (i+2)^{\sharp 1}\} & (0 \le i \le w-2), \\
\{i^{\sharp 2}, (i+3)^{\sharp 1}\} & (0 \le i \le w-3), \\
\{i^{\sharp 2}, (i+4)^{\sharp 2}, (i+2)^{\sharp 1}\} & (0 \le i \le w-4), \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+3)^{\sharp 1}\} & (0 \le i \le w-3), \\
\{i^{\sharp 2}, (i+2)^{\sharp 1}, (i+3)^{\sharp 1}\} & (0 \le i \le w-3), \\
\{i^{\sharp 1}, (i+3)^{\sharp 2}\} & (0 \le i \le w-3), \\
\{i^{\sharp 1}, (i+2)^{\sharp 2}\} & (0 \le i \le w-2), \\
\{i^{\sharp 1}, (i+1)^{\sharp 1}, (i+3)^{\sharp 2}\} & (0 \le i \le w-3), \\
\{i^{\sharp 1}, (i+2)^{\sharp 2}, (i+3)^{\sharp 2}\} & (0 \le i \le w-3), \\
\{i^{\sharp 2}\} & (0 \le i \le w), \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}\} & (0 \le i \le w-1), \\
\{i^{\sharp 2}, (i+2)^{\sharp 2}\} & (0 \le i \le w-2), \\
\{i^{\sharp 2}, (i+3)^{\sharp 2}\} & (0 \le i \le w-3), \\
\{i^{\sharp 2}, (i+4)^{\sharp 2}\} & (0 \le i \le w-4), \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+2)^{\sharp 2}\} & (0 \le i \le w-2),
\end{array}
$$

$$
\begin{aligned}
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+3)^{\sharp 2}\} &\qquad (0 \leq i \leq w-3), \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+4)^{\sharp 2}\} &\qquad (0 \leq i \leq w-4), \\
\{i^{\sharp 2}, (i+2)^{\sharp 2}, (i+3)^{\sharp 2}\} &\qquad (0 \leq i \leq w-3), \\
\{i^{\sharp 2}, (i+2)^{\sharp 2}, (i+4)^{\sharp 2}\} &\qquad (0 \leq i \leq w-4), \\
\{i^{\sharp 2}, (i+3)^{\sharp 2}, (i+4)^{\sharp 2}\} &\qquad (0 \leq i \leq w-4) \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+2)^{\sharp 2}, (i+3)^{\sharp 2}\} &\qquad (0 \leq i \leq w-3), \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+2)^{\sharp 2}, (i+4)^{\sharp 2}\} &\qquad (0 \leq i \leq w-4), \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+3)^{\sharp 2}, (i+4)^{\sharp 2}\} &\qquad (0 \leq i \leq w-4), \\
\{i^{\sharp 2}, (i+2)^{\sharp 2}, (i+3)^{\sharp 2}, (i+4)^{\sharp 2}\} &\qquad (0 \leq i \leq w-4), \\
\{i^{\sharp 2}, (i+1)^{\sharp 2}, (i+2)^{\sharp 2}, (i+3)^{\sharp 2}, (i+4)^{\sharp 2}\} &\qquad (0 \leq i \leq w-4).
\end{aligned}
$$

Chapter 6

String correction using imitation of Levensthein-automata

We now introduce a variant of the correction method described in Chapter 3. The main advantage of the new method is that it avoids the actual computation of Levenshtein-automata. As before we assume that for some fixed degree n we have at our disposal a generic description of the automaton $LEV_n(W) = \langle \Sigma, Q^W, \{0^{\sharp 0}\}, F^W, \Delta^W \rangle$ for arbitrary input W, as presented in the previous chapter for degree $n = 1$. With Δ_χ^W we denote the variant of the the transition function where characteristic vectors are treated as input. Note that the tables T_n yield parametric descriptions of Δ_χ^W. For example, from table T_1 we see that A_0 maps to C_0 under vector $\langle 0, 0, 0 \rangle$.

As in Chapter 3 we assume that the dictionary is implemented in the form of a deterministic finite state automaton $A_D = \langle \Sigma, Q^D, q_0^D, F^D, \delta^D \rangle$.

Given any concrete input word W, we first compute the set of all characteristic vectors of the form $\chi(x, W_{[i]})$ where $x \in \Sigma$ and i denotes a boundary of W. Using this vectors, the backtracking procedure given in Chapter 3 can now be replaced by the following variant:

```
push (<\varepsilon, q_0^D, \{0^{\sharp 0}\}>);
while not empty(stack) do begin
   pop (<V, q^D, M>);
   for x in \Sigma do begin
      q_1^D := \delta^D(q^D, x);
      M' := \Delta_\chi^W(M, \chi(x, W_{[M]}));
      if (q_1^D <> NIL) and (M' <> NIL) then begin
         V_1 := concat(V, x);
         push(<V_1, q_1^D, M'>);
         if (q_1^D \in F^D) and (M' \in F^W) then output(V_1);
      end;
   end;
end;
```

Note that in contrast to the situation described in Chapter 3, we do not assume that the Levenshtein-automaton for the concrete input word W is available. Given the generic description of LEV_n, states of $LEV_n(W)$ are only introduced on demand in line 6. It is important to note that each image state $\Delta_\chi^W(M, \chi(x, W_{[M]}))$ can be found in constant time since both $\chi(x, W_{[M]})$ and the table T_n for Δ_χ have been precomputed. The following example illustrates the modified acceptance procedure.

Example 6.0.5 We consider the case $n = 1$. Assume that the (misspelled) input word W has the form "chold". We consider the path of the dictionary automaton for the dictionary entry "child", which is assumed to lead to a final state. The following transition sequence illustrates how states of $LEV_1(chold)$ are generated on demand, using precomputed characteristic vectors and Table 5.2.

$$\begin{array}{rl}
A_0 \text{ input } \chi(c, cho) = \langle 1,0,0\rangle & \mapsto A_1 \\
A_1 \text{ input } \chi(h, hol) = \langle 1,0,0\rangle & \mapsto A_2 \\
A_2 \text{ input } \chi(i, old) = \langle 0,0,0\rangle & \mapsto C_2 \\
C_2 \text{ input } \chi(l, old) = \langle 0,1,0\rangle & \mapsto B_4 \\
B_4 \text{ input } \chi(d, d) = \langle 1\rangle & \mapsto B_5
\end{array}$$

Now B_5 is a final state. Hence, in the above procedure, the word "child" is suggested as one correction of the input "chold". Assume now that the dictionary also contains the word "cold". In this case we reach the following states of $LEV_1(chold)$:

$$\begin{array}{rl}
A_0 \text{ input } \chi(c, cho) = \langle 1,0,0\rangle & \mapsto A_1 \\
A_1 \text{ input } \chi(o, hol) = \langle 0,1,0\rangle & \mapsto E_1 \\
E_1 \text{ input } \chi(l, hol) = \langle 0,0,1\rangle & \mapsto B_4 \\
B_4 \text{ input } \chi(d, d) = \langle 1\rangle & \mapsto B_5
\end{array}$$

Since B_5 is final, also "cold" is suggested as a correction candidate.

Chapter 7

Adding Transpositions

As a matter of fact, the structure of Levenshtein-automata is affected if further primitive edit operations are used. In this chapter we consider the situation where insertions, deletions, substitutions and *transpositions* are treated as primitive edit operations. The methodology for defining states and transitions presented in Chapter 4 can be extended.

7.1 A family of deterministic Levenshtein-automata for primitive edit operations including transpositions

Motivated by the intended application of the correction methods in a typesetting context we assume that primitive edit operatios are applied in parallel. This means that if V is obtained from $W = x_1 \cdots x_w$ with a transposition of the letters $x_{i+1}x_{i+2}$, then V has an occurrence of the transposed sequence $x_{i+2}x_{i+1}$.[1] In the sequel, with Levenshtein-distance we always mean the distance where transpositions are treated as primitive edit operations. With $D_L^T(V, W)$ we denote the distance between V and W, and $\mathcal{L}_{Lev}^T(n, W)$ denotes the set of all words $V \in \Sigma^*$ such that $d_L^T(W, V) \leq n$.

Since primitive edit operations are applied in parallel, the concept of a trace representation can be extended. Each transposition is represented by means of a pair of crossing strokes. Endpoints are neighboured letters.

Remark 7.1.1 Let $W = x_1x_2 \cdots x_w$ and $V = y_1y_2 \cdots y_v$ be two words with Levenshtein-distance $n \geq 1$. Assume that neither V is a prefix of W nor vice versa. Let $U = x_1x_2 \cdots x_i$ (where $0 \leq i \leq v, w$) denote the maximal common prefix of V and W. Then, in any trace representation of a minimal sequence ν of edit operations leading from W to V exactly one of the following four cases holds:

1.-3. *Insertion, substitution, deletion cases.* As in Remark 2.0.3.

[1] When using transpositions, the question if operations are applied sequentially or in parallel is in fact relevant. Our assumption implies, for example, that the words "*ab*" and "*bca*" have distance 3.

4. *Transposition case.* x_{i+1} and x_{i+2} are respectively connected with y_{i+2} and y_{i+1}, strokes are crossing.

Let $W = x_1 \cdots x_w$ denote the input word and let n denote the degree. As before, the elements of $\{0, 1, \ldots, w\}$ are called *boundaries* of W.

Definition 7.1.2 A *standard position* is an expression of the form $i^{\sharp e}$ where i is a boundary and $0 \leq e \leq n$. A *t-position* is an expression of the form $i_t^{\sharp e}$ where $0 \leq i \leq w-2$ and $1 \leq e \leq n$. A *position* is either a standard position or a t-position. A position π is an *accepting position* iff $\pi = i^{\sharp e}$ is a standard position and if $w - i \leq n - e$.

The intuitive interpretation of expressions $i^{\sharp e}$ is as before. Expressions $i_t^{\sharp e}$ are reached from positions $i^{\sharp e-1}$ under input x_{i+2} in situations where we have a transposition of the letters x_{i+1} and x_{i+2}.

Definition 7.1.3 Subsumption between positions is explained as follows:

1. A position $i^{\sharp e}$ *subsumes* a position $j^{\sharp f}$ iff $e < f$ and $|j - i| \leq f - e$.
2. A position $i^{\sharp e}$ *subsumes* a position $j_t^{\sharp f}$ iff $f > e$ and $|j - (i-1)| \leq f - e$.
3. A position $i_t^{\sharp e}$ *subsumes* a position $j^{\sharp f}$ iff $n = f > e$ and $i = j$.
4. A position $i_t^{\sharp e}$ *subsumes* a position $j_t^{\sharp f}$ iff $f > e$ and $i = j$.

As in Chapter 4, with each position π we associate a language $\Phi(\pi)$.

$$\begin{array}{rcl} \Phi(i^{\sharp e}) & := & \mathcal{L}^T_{Lev}(n-e, x_{i+1} \cdots x_w), \\ \Phi(i_t^{\sharp e}) & := & \{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-e, x_{i+3} \cdots x_w). \end{array}$$

Lemma 7.1.4 *Let π and π' denote two distinct positions. If π subsumes π', then $\Phi(\pi') \subseteq \Phi(\pi)$.*

Definition 7.1.5 Let $0 \leq i \leq w$. A *state with base position $i^{\sharp 0}$* is a set M of positions, not necessarily containing $i^{\sharp 0}$, that safisfies the following properties:

1. for each position $j^{\sharp e}$ or $j_t^{\sharp e}$ in M we have $|i - j| \leq e$. In addition, for each t-position $j_t^{\sharp e}$ we have $|j - (i-1)| \leq e$,
2. M does not contain any position that is subsumed by another element of M.

Definition 7.1.6 Let $W = x_1 \cdots x_w$ as above. Let $\pi := i^{\sharp e}$ be a position, and let $k := \min\{n-e+1, w-i\}$. The *relevant subword of W for position π*, denoted $W_{[\pi]}$, is the subword $x_{i+1} \cdots x_{i+k}$ of W. Now let $i \leq w-2$. The *relevant subword of W for position $\pi := i_t^{\sharp e}$*, denoted $W_{[\pi]}$, is the subword $x_{i+1} \cdots x_{i+k}$ of W.

Definition 7.1.7 Let W and n as above. An *elementary transition* assigns to each position π and each symbol $x \in \Sigma$ a state $\delta(\pi, x)$. The complete set of elementary transitions is specified in Table 7.1.

Lemma 7.1.8 *Let M be a state with base position $i^{\sharp 0}$ where $i < w$. Then the image of any element of M under an elementary transition is always a state with base position $(i+1)^{\sharp 0}$. If M is a state with base position $w^{\sharp 0}$, then the image of any element of M under an elementary transition is always a state with base position $w^{\sharp 0}$.*

Proof. Let $i < w$, assume that $i^{\sharp 0}$ subsumes a t-position $j_t^{\sharp f} \in M$. Then $i - e \leq j \leq i + e - 1$. The only non-empty possible image of $j_t^{\sharp f}$ is $\{(j+2)^{\sharp f}\}$. We have $i + 1 - e \leq j + 2 \leq i + 1 + e$. It follows that $(i+1)^{\sharp e}$ subsumes $(j+2)^{\sharp f}$. The remaining cases are straightforward. □

The proof of the following lemma is straightforward.

Lemma 7.1.9 (Raising Lemma for elementary transitions) *Let $n > 0$ and $1 \leq e \leq n$. Then for any position of degree n of the form $[\pi]^e$ and any $x \in \Sigma$ we have*

$$\delta^{(n)}([\pi]^e, x) = [\delta^{(n-e)}(\pi, x)]^{\sharp e}.$$

In the sequel, with $M \sqcup N$ we denote the reduced union of the states M and N with common base position where we refer to the new notion of subsumption introduced in Definition 7.1.3.

Definition 7.1.10 Let $W = x_1 \cdots x_w$ where $w \geq 0$, let $n \geq 0$. Then $LEV_n^T(W)$ is the deterministic finite state automaton $\langle \Sigma, Q, q_0, F, \Delta \rangle$ where

1. the set of *states* Q contains all states in the sense of Definition 7.1.5,
2. the *initial state* is $q_0 := \{0^{\sharp 0}\}$,
3. the set F of *final states* contains all states $M \in Q$ that contain an accepting position,
4. the *transition function* Δ is defined in the following way: for any symbol $y \in \Sigma$ and any state $M \in Q$, $\Delta(M, y) := \bigsqcup_{\pi \in M} \delta(\pi, y)$.

It follows from Lemma 7.1.8 that Δ assigns to each state $M \in Q$ and each $y \in \Sigma$ again a state $M \in Q$. In fact, a state with base position $i^{\sharp 0}$ $(i < w)$ is always mapped to a state with base position $(i+1)^{\sharp 0}$, and states with base position $w^{\sharp 0}$ are mapped to states with base position $w^{\sharp 0}$. This shows that $LEV_n^T(W)$ is a deterministic finite state automaton.

The raising lemma holds also in the new situation. We write $\Delta^{(n)}$ for the transition function of $LEV_n^T(W)$.

Lemma 7.1.11 (Raising Lemma for transitions) *Let $n > 0$ and $1 \leq e \leq n$. Then for any state of degree n of the form $[M]^{\sharp e}$ and any $x \in \Sigma$ we have*

$$\Delta^{(n)}([M]^{\sharp e}, x) = [\Delta^{(n-e)}(M, x)]^{\sharp e}.$$

Proof. As for Lemma 4.0.29. □

In the sequel, let $LEV_n^T(W) = \langle \Sigma, Q, q_0, F, \Delta \rangle$ as in Definition 7.1.10.

Proposition 7.1.12 *The following properties hold:*

1. $\mathcal{L}(\emptyset) = \emptyset$,
2. *for all states* M, N *with a common base position and all* $y \in \Sigma$:
$$\Delta(M \sqcup N, y) = \Delta(M, y) \sqcup \Delta(N, y),$$
3. *for all states* M, N *with a common base position and all* $V \in \Sigma^*$:
$$\Delta^*(M \sqcup N, V) = \Delta^*(M, V) \sqcup \Delta^*(N, V),$$
4. *for all states* $M \subseteq Q \setminus \{\{0^{\sharp 0}\}, \ldots, \{w^{\sharp 0}\}\}$: $\mathcal{L}(M) = \bigcup_{\pi \in M} \mathcal{L}(\{\pi\})$.

Proof. As for Proposition 4.0.30. □

Proposition 7.1.13 *The following holds:*

$$\begin{array}{rcl} \mathcal{L}(\{i_t^{\sharp e}\}) & = & \{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-e, x_{i+3} \cdots x_w) \quad (0 \leq i \leq w-2, 1 \leq e \leq n) \\ \mathcal{L}(\{i^{\sharp e}\}) & = & \mathcal{L}^T_{Lev}(n-e, x_{i+1} \cdots x_w) \quad (0 \leq i \leq w, 0 \leq e \leq n) \end{array}$$

Proof. We proceed by induction on n. The case $n = 0$ can be treated as before (cf. proof of Prop. 4.0.31).

Now let $n \geq 1$ and assume that the Proposition is correct for all $0 \leq n' < n$. The case $e = n$ is simple since all relevant transitions are of the form $\{i^{\sharp n}\} \mapsto \{(i+1)^{\sharp n}\}$ under x_{i+1} $(i < w)$ or $\{i_t^{\sharp n}\} \mapsto \{(i+2)^{\sharp n}\}$ under x_{i+1} $(i < w-1)$. Hence assume that $e < n$. Since for $1 \leq e < n$ transitions from states $\{i^{\sharp e}\}$ are defined by raising of transitions of degree $n' = n - e$ the induction hypothesis shows that

$$\mathcal{L}(\{i_t^{\sharp e}\}) = \{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-e, x_{i+3} \cdots x_w) \quad (0 \leq i \leq w-2, 2 \leq e \leq n) \tag{7.1}$$
$$\mathcal{L}(\{i^{\sharp e}\}) = \mathcal{L}^T_{Lev}(n-e, x_{i+1} \cdots x_w) \quad (0 \leq i \leq w, 1 \leq e \leq n) \tag{7.2}$$

It remains to prove that

$$\mathcal{L}(\{i_t^{\sharp 1}\}) = \{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-1, x_{i+3} \cdots x_w) \quad (0 \leq i \leq w-2) \tag{7.3}$$
$$\mathcal{L}(\{i^{\sharp 0}\}) = \mathcal{L}^T_{Lev}(n, x_{i+1} \cdots x_w) \quad (0 \leq i \leq w) \tag{7.4}$$

In the sequel, let W_i denote the suffix $x_{i+1} \cdots x_w$ of W $(0 \leq i \leq w)$. From (7.2) and Lemma 7.1.4 we obtain the following : for all positions $i^{\sharp e}$ and $j^{\sharp f}$ such that $e \neq 0 \neq f$: if $i^{\sharp e}$ is subsumed by $j^{\sharp f}$, then $\mathcal{L}(\{i^{\sharp e}\})$ is a proper subset of $\mathcal{L}(\{j^{\sharp f}\})$ (††).

I. We first show that $\{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-1, x_{i+3} \cdots x_w) \subseteq \mathcal{L}(\{i_t^{\sharp 1}\})$. The elementary transitions show that from $\{i_t^{\sharp 1}\}$ we reach $\{(i+2)^{\sharp 1}\}$ consuming x_{i+1}. Now (2) shows that $\{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-1, x_{i+3} \cdots x_w) \subseteq \mathcal{L}(\{i_t^{\sharp 1}\})$.

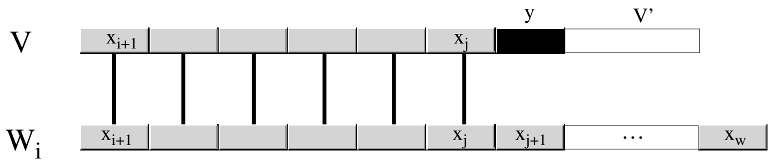
II. We show that $\mathcal{L}(\{i_t^{\sharp 1}\}) \subseteq \{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-1, W_{i+2})$. The only possibility to proceed from $\mathcal{L}(\{i_t^{\sharp 1}\})$ is using input x_{i+1} to reach $\{(i+2)^{\sharp 1}\}$. By (2), $\mathcal{L}(\{(i+2)^{\sharp e}\}) = \mathcal{L}^T_{Lev}(n-1, x_{i+3} \cdots x_w)$. It follows that $\mathcal{L}(\{i_t^{\sharp 1}\}) \subseteq \{x_{i+1}\} \circ \mathcal{L}^T_{Lev}(n-1, W_{i+2})$.

III. We show that $\mathcal{L}^T_{Lev}(n, W_i) \subseteq \mathcal{L}(\{i^{\sharp 0}\})$. Let $V \in \mathcal{L}^T_{Lev}(n, W_i)$.

Case 1.1: V is obtained from W_i by deleting a suffix of length k $(0 \leq k \leq n)$ of W_i. Starting from state $\{i^{\sharp 0}\}$ and consuming V we reach state $\{(w-k)^{\sharp 0}\}$. Since $(w-k)^{\sharp 0}$ is an accepting position, state $\{(w-k)^{\sharp 0}\}$ is final, hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

Case 1.2: W_i is obtained from V by deleting a suffix of length k $(1 \leq k \leq n)$ of V. Starting from state $\{i^{\sharp 0}\}$ and first consuming W_i we reach $\{w^{\sharp 0}\}$. The k additional transitions lead to $\{w^{\sharp k}\}$. Since $w^{\sharp k}$ is an accepting position, the latter state is final, hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

Case 2: In the remaining cases there exists an index $j < w$ such that $V = x_{i+1} \cdots x_j y V'$ where $y \neq x_{j+1}$.



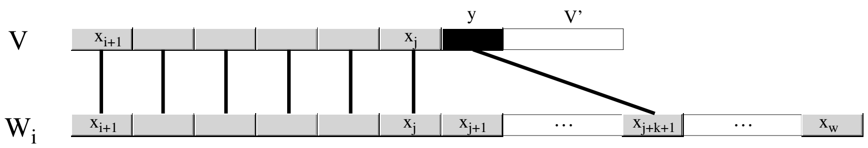
We have $yV' \in \mathcal{L}^T_{Lev}(n, x_{j+1} \cdots x_w)$ by Lemma 2.0.2. We fix a sequence ν of edit operations leading from $x_{j+1} \cdots x_w$ to yV' of minimal length and consider the four cases described in Remark 7.1.1.

2.1. If the occurrence of y is an insertion before x_{j+1} (where $i \leq j \leq w$), then $V' \in \mathcal{L}^T_{Lev}(n-1, x_{j+1} \cdots x_w)$. Since $y \neq x_{j+1}$, starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots, \{j^{\sharp 0}\}, M$ where M contains $j^{\sharp 1}$ (cf. elementary transitions). It follows from (2) and Part 4 of Proposition 4.0.30 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.2. If the occurrence of y substitutes x_{j+1} (where $i \leq j < w$), then V' belongs to $\mathcal{L}^T_{Lev}(n-1, x_{j+2} \cdots x_w)$. Starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots, \{j^{\sharp 0}\}, M$ where M contains $(j+1)^{\sharp 1}$. It follows from (2) and Part 4 of Proposition 7.1.12 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.3. If the occurrence of $y = x_{j+2}$ is caused by a transposition of x_{j+1} and x_{j+2} (where $j \leq w-2$), then $V' = x_{j+1} V''$ where $V'' \in \mathcal{L}^T_{Lev}(n-1, x_{j+3} \cdots x_w)$. Since $y \neq x_{j+1}$, starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, x_{j+2}$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots, \{j^{\sharp 0}\}, M$ where M contains $j_t^{\sharp 1}$ (cf. elementary transitions). From M consuming x_{j+1} we reach a state M' with a position identical to or subsuming $(j+2)^{\sharp 1}$. It is easy to see that all positions of M' must be raised, hence M' contains $(j+2)^{\sharp 1}$. From part II we see that $V'' \in \mathcal{L}(M')$. It follows that $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.4. In the remaining case there exists some $1 \leq k \leq n$ such that we have a stroke from x_{j+k+1} to y in the trace representation of ν.



This means that the k letters $x_{j+1}, x_{j+2}, \ldots, x_{j+k}$ are erased. The distance between $x_{j+k+1} \cdots x_w$ and yV' is bounded by $n-k$. In the sequel, let k_0 be the smallest index in $\{1, \ldots, k+1\}$ such that $x_{j+k+1} = x_{j+k_0}$. It follows from the definition of elementary transitions (cf. Table 4.1) that we reach state $M := \{j^{\sharp 1}, (j+1)^{\sharp 1}, (j+k_0)^{\sharp k_0 - 1}\}$ from $\{j^{\sharp 0}\}$ by consuming $x_{j+k_0} = x_{j+k+1}$.

2.4.1. Assume first that $x_{j+k+1} = y$. Then the distance between $x_{j+k+2} \cdots x_w$ and V' is bounded by $n-k$. By (2), $V' \in \mathcal{L}(\{(j+k+1)^{\sharp k}\})$. Since $(j+k+1)^{\sharp k}$ is subsumed by $(j+k_0)^{\sharp k_0 - 1}$ also $V' \in \mathcal{L}(\{(j+k_0)^{\sharp k_0 - 1}\})$, by (††) and $V' \in \mathcal{L}(M)$ by Part 4 of Proposition 4.0.30. It follows that $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.4.2. Assume that $x_{j+k+1} \neq y$. Then the distance between $x_{j+k+2} \cdots x_w$ and V' is bounded by $n-k-1$. Starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we eventually reach a state M containing $(j+1)^{\sharp 1}$. This position subsumes $(j+k+1)^{\sharp k+1}$. It follows from (2), (††) and Part 4 of Proposition 4.0.30 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

IV. It remains to prove that $\mathcal{L}(\{i^{\sharp 0}\}) \subseteq \mathcal{L}^T_{Lev}(n, W_i)$ for $0 \leq i \leq w$. Let $V \in \mathcal{L}(\{i^{\sharp 0}\})$. If V is accepted - starting from $\{i^{\sharp 0}\}$ - on a path of singleton sets with basic positions $\{(i+1)^{\sharp 0}\}, \{(i+2)^{\sharp 0}\}, \ldots, \{k^{\sharp 0}\}$, then $k^{\sharp 0}$ is accepting which implies that V has the form $x_{i+1} \cdots x_k$ where $w - k \leq n$. This shows that $V \in \mathcal{L}^T_{Lev}(n, W_i)$. In the other case, starting from state $\{i^{\sharp 0}\}$ and consuming the prefix $V'y$ of $V = V'yV''$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots \{j^{\sharp 0}\}, M$ where $M \neq \{(j+1)^{\sharp 0}\}$.

Case (a): $j < w$ and M has the form $\{j^{\sharp 1}, (j+1)^{\sharp 1}\}$. In this case, by (2) and Part 4 of Proposition 4.0.30, either V'' has distance $\leq n-1$ to $x_{j+1} \cdots x_w$ or V'' has distance $\leq n-1$ to $x_{j+2} \cdots x_w$. In the former case, with an additional insertion of y we see that V has distance $\leq n$ to W_i. In the latter case, using a substitution $x_{j+1} \mapsto y$ we see V has distance $\leq n$ to W_i.

Case (b): $j < w$ and M has the form $\{j^{\sharp 1}, (j+1)^{\sharp 1}, (j+k)^{\sharp k-1}\}$. Here we have to consider the additional case where V'' has distance $\leq n-(k-1)$ to $x_{j+k+1} \cdots x_w$. However, we know that $y = x_{j+k}$. Deleting $x_{j+1}, \ldots, x_{j+k-1}$ we see that yV' has distance $\leq n$ to $x_{j+1} \ldots x_w$, hence the same holds for V and W_i.

Case (c): $j < w-1$ and M has the form $\{j^{\sharp 1}, j_t^{\sharp 1}, (j+1)^{\sharp 1}, (j+k)^{\sharp k-1}\}$. This is similar to (b), but $y = x_{j+2}$ and we have the possibility that $V' = x_{j+1}V''$ where V'' has distance $\leq n-1$ to $x_{j+3} \cdots x_w$. Here $x_{j+2}x_{j+1}V''$ has distance $\leq n$ to $x_{j+1} \cdots x_w$, hence V has distance $\leq n$ to W_i.

Case (d): $j = w$. In this case $M = \{w^{\sharp 1}\}$ and $V' = W_i$. It follows from (2) that V'' has distance $\leq n-1$ to the empty word ε. Hence V has distance $\leq n$ to W_i. □

From Proposition 7.1.13 we obtain the parallel result to Theorem 4.0.32.

Theorem 7.1.14 *$LEV^T_n(W)$ is a deterministic and acyclic Levenshtein-automaton of degree n for W for primitive edit operations including transpositions. For fixed degree n, the size of $LEV^T_n(W)$ is linear in $|W|$.*

7.2 Computation of deterministic Levenshtein-automata for primitive edit operations including transpositions

As for $LEV_n(W)$, the general description of the Levensthein-automaton $LEV_n^T(W)$ can be used to derive, for any fixed number n, an algorithm that actually computes the automaton $LEV_n^T(W)$ in linear time, given any input word W. The principle will be illustrated for degree $n = 1$.

For input $W = x_1 \cdots x_w$ and $n = 1$ the list of positions is

$$\begin{array}{l} 0^{\sharp 0}, \ldots, w^{\sharp 0}, 0^{\sharp 1}, \ldots, w^{\sharp 1} \\ 0_t^{\sharp 1}, \ldots, (w-2)_t^{\sharp 1}. \end{array}$$

It suffices to consider the following states:

$$\begin{array}{rcl} \emptyset & & \text{failure state}, \\ A_i & := & \{i^{\sharp 0}\} \quad (0 \le i \le w), \\ B_i & := & \{i^{\sharp 1}\} \quad (0 \le i \le w), \\ C_i & := & \{i^{\sharp 1}, (i+1)^{\sharp 1}\} \quad (0 \le i \le w-1), \\ D_i & := & \{i^{\sharp 1}, (i+2)^{\sharp 1}\} \quad (0 \le i \le w-2), \\ E_i & := & \{i^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \quad (0 \le i \le w-2), \\ F_i & := & \{i^{\sharp 1}, i_t^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \quad (0 \le i \le w-2), \end{array}$$

The initial state is A_0. Accepting positions are $w^{\sharp 1}$, $w^{\sharp 0}$, as well as $(w-1)^{\sharp 0}$ for $w \ge 1$. It follows immediately that the final states are

$$\begin{array}{rl} A_w, A_{w-1}, B_w, C_{w-1}, D_{w-2}, E_{w-2}, F_{w-2} & \quad \text{for } w \ge 2, \\ A_w, A_{w-1}, B_w, C_{w-1} & \quad \text{for } w = 1, \\ A_w, B_w & \quad \text{for } w = 0. \end{array}$$

For the case $n = 1$, we obtain the set of elementary transitions given in Table 7.2. Using this Table 7.2 it is simple to compute a parametric description of the full transition function Δ as before. The description is given in Table 7.3.

Obviously, given the above generic description of LEV_1^T it is possible to generate for any concrete input W the automaton $LEV_1^T(W)$ in time $O(|W|)$.

Theorem 7.2.1 *There exists an algorithm that computes for any input word W the automaton $LEV_1^T(W)$ in time and space $O(|W|)$.*

Corollary 7.2.2 *For any input W, the minimal deterministic Levenshtein-automaton of degree 1 for W where primitive edit operations include transpositions can be computed in time and space $O(|W|)$.*

Example 7.2.3 Figure 7.1 describes the automaton $LEV_1^T(W)$ for the input word "atlas". For each word W of length 5 with 3-profile sequence $(1,2,3),(1,2,3),(1,2,3)$ the automaton $LEV_1^T(W)$ has the same structure, modulo transition labels. Similarly Figure 7.2 describes the structure of $LEV_1^T(W)$ for the word "otter". Here for each word W of length 5 with 3-profile sequence $(1,2,2),(1,1,2),(1,2,3)$ the automaton $LEV_1^T(W)$ has the same structure, modulo transition labels.

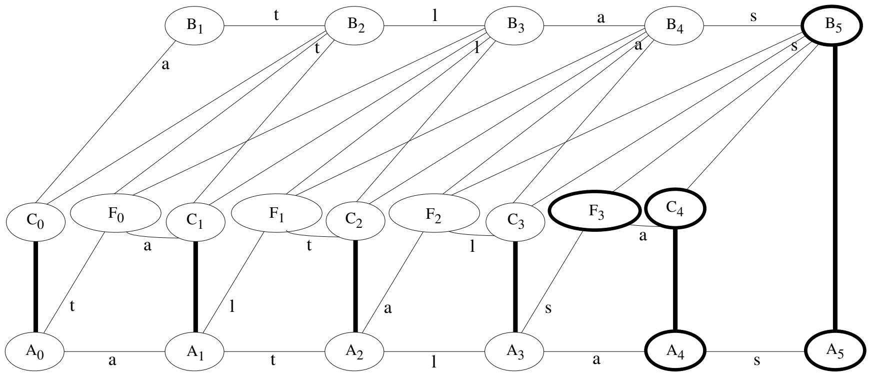


Figure 7.1: Deterministic Levenshtein-automaton $LEV_1^T(W)$ for input W = “atlas”.

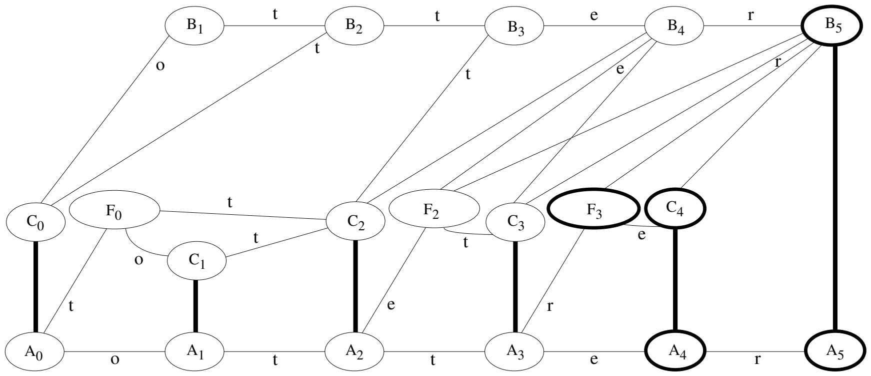


Figure 7.2: Deterministic Levenshtein-automaton $LEV_1^T(W)$ for input W = “otter”.

As a matter of fact, the same method can be used for any fixed degree n.

Theorem 7.2.4 *For any fixed degree n, there exists an algorithm that computes for any input word W the automaton $LEV_n^T(W)$ in time and space $O(|W|)$.*

Corollary 7.2.5 *For any input W, the minimal deterministic Levenshtein-automaton of degree n for W where primitive edit operations include transpositions can be computed in time and space $O(|W|)$.*

(I) $e = 0 < n$		
$i \le w - 2$	$\delta(i^{\sharp 0}, x) :=$	$\begin{cases} \{(i+1)^{\sharp 0}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 1, b_2, \ldots, b_k \rangle, \\ \{i^{\sharp 1}, i_t^{\sharp 1}, (i+1)^{\sharp 1}, (i+j)^{\sharp j-1}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 0, 1, b_3, \ldots, b_k \rangle, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}, (i+j)^{\sharp j-1}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 0, 0, b_2, \ldots, b_k \rangle : j, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 0, \ldots, 0 \rangle. \end{cases}$
$i = w - 1$	$\delta(i^{\sharp 0}, x) :=$	$\begin{cases} \{(i+1)^{\sharp 0}\} & \text{for } \chi(x, x_{i+1}) = \langle 1 \rangle, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}\} & \text{for } \chi(x, x_{i+1}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp 0}, x) := \{w^{\sharp 1}\}$	
(II) $1 \le e \le n - 1$		
$i \le w - 2$	$\delta(i^{\sharp e}, x) :=$	$\begin{cases} \{(i+1)^{\sharp e}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 1, b_2, \ldots, b_k \rangle, \\ \{i^{\sharp e+1}, i_t^{\sharp e+1}, (i+1)^{\sharp e+1}, (i+j)^{\sharp e+j-1}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 0, 1, b_3, \ldots, b_k \rangle, \\ \{i^{\sharp e+1}, (i+1)^{\sharp e+1}, (i+j)^{\sharp e+j-1}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 0, 0, b_2, \ldots, b_k \rangle : j, \\ \{i^{\sharp e+1}, (i+1)^{\sharp e+1}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 0, \ldots, 0 \rangle. \end{cases}$
	$\delta(i_t^{\sharp e}, x) :=$	$\begin{cases} \{(i+2)^{\sharp e}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 1, b_2 \ldots, b_k \rangle, \\ \emptyset & \text{else.} \end{cases}$
$i = w - 1$	$\delta(i^{\sharp e}, x) :=$	$\begin{cases} \{(i+1)^{\sharp e}\} & \text{for } \chi(x, x_{i+1}) = \langle 1 \rangle, \\ \{i^{\sharp e+1}, (i+1)^{\sharp e+1}\} & \text{for } \chi(x, x_{i+1}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp e}, x) := \{w^{\sharp e+1}\}$	
(III) $e = n$		
$i \le w - 1$	$\delta(i^{\sharp n}, x) :=$	$\begin{cases} \{(i+1)^{\sharp n}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 1 \rangle, \\ \emptyset & \text{for } \chi(x, W_{[\pi]}) = \langle 0 \rangle. \end{cases}$
$i \le w - 2$	$\delta(i_t^{\sharp n}, x) :=$	$\begin{cases} \{(i+2)^{\sharp n}\} & \text{for } \chi(x, W_{[\pi]}) = \langle 1 \rangle, \\ \emptyset & \text{for } \chi(x, W_{[\pi]}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp n}, x) := \emptyset.$	

Table 7.1: Table of elementary transitions for $\pi = i^{\sharp e}$ resp. $\pi = i_t^{\sharp e}$ where transpositions are treated as primitive edit operations.

(I) $e = 0$	
$i \leq w-2$	$\delta(i^{\sharp 0}, x) := \begin{cases} \{(i+1)^{\sharp 0}\} \\ \quad \text{for } \chi(x, x_{i+1}x_{i+2}) = \langle 1, b_2\rangle, \\ \{i^{\sharp 1}, i_t^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \\ \quad \text{for } \chi(x, x_{i+1}x_{i+2}) = \langle 0, 1\rangle, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0, 0\rangle. \end{cases}$
$i \leq w-1$	$\delta(i^{\sharp 0}, x) := \begin{cases} \{(i+1)^{\sharp 0}\} \\ \quad \text{for } \chi(x, x_{i+1}) = \langle 1\rangle, \\ \{i^{\sharp 1}, (i+1)^{\sharp 1}\} \\ \quad \text{for } \chi(x, x_{i+1}) = \langle 0\rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp 0}, x) := \{w^{\sharp 1}\}$
(II) $e = 1$	
$i \leq w-2$	$\delta(i^{\sharp 1}, x) := \begin{cases} \{(i+1)^{\sharp 1}\} \quad \text{for } \chi(x, x_{i+1}) = \langle 1\rangle, \\ \emptyset \quad \text{for } \chi(x, x_{i+1}) = \langle 0\rangle. \end{cases}$ $\delta(i_t^{\sharp 1}, x) := \begin{cases} \{(i+2)^{\sharp 1}\} \quad \text{for } \chi(x, x_{i+1}) = \langle 1\rangle, \\ \emptyset \quad \text{for } \chi(x, x_{i+1}) = \langle 0\rangle. \end{cases}$
$i \leq w-1$	$\delta(i^{\sharp 1}, x) := \begin{cases} \{(i+1)^{\sharp 1}\} \quad \text{for } \chi(x, x_{i+1}) = \langle 1\rangle, \\ \emptyset \quad \text{for } \chi(x, x_{i+1}) = \langle 0\rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp 1}, x) := \emptyset.$

Table 7.2: Table of elementary transitions for $n = 1$, primitive edit operations including transpositions.

$0 \leq i \leq w-3$						
$\chi(x, x_{i+1}x_{i+2}x_{i+3})$	A_i	B_i	C_i	D_i	E_i	F_i
$\langle 0,0,0\rangle$	C_i	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\langle 1,0,0\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	C_{i+1}
$\langle 0,1,0\rangle$	F_i	$\emptyset$	B_{i+2}	$\emptyset$	B_{i+2}	B_{i+2}
$\langle 0,0,1\rangle$	C_i	$\emptyset$	$\emptyset$	B_{i+3}	B_{i+3}	B_{i+3}
$\langle 1,1,0\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	B_{i+1}	C_{i+1}	C_{i+1}
$\langle 1,0,1\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	D_{i+1}	D_{i+1}	E_{i+1}
$\langle 0,1,1\rangle$	F_i	$\emptyset$	B_{i+2}	B_{i+3}	C_{i+2}	C_{i+2}
$\langle 1,1,1\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	D_{i+1}	E_{i+1}	E_{i+1}

$i = w-2$						
$\chi(x, x_{i+1}x_{i+2})$	A_i	B_i	C_i	D_i	E_i	F_i
$\langle 0,0\rangle$	C_i	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\langle 1,0\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	C_{i+1}
$\langle 0,1\rangle$	F_i	$\emptyset$	B_{i+2}	$\emptyset$	B_{i+2}	B_{i+2}
$\langle 1,1\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	B_{i+1}	C_{i+1}	C_{i+1}

$i = w-1$			
$\chi(x, x_{i+1})$	A_i	B_i	C_i
$\langle 0\rangle$	C_i	$\emptyset$	$\emptyset$
$\langle 1\rangle$	A_{i+1}	B_{i+1}	B_{i+1}

$i = w$		
$\chi(x, \varepsilon)$	A_i	B_i
$\langle\rangle$	B_i	$\emptyset$

Table 7.3: Transitions of $LEV_1^T(W)$.

Chapter 8

Adding Merges and Splits

The last case that we consider is the situation where insertions, deletions, substitutions, *merges* and *splits* are treated as primitive edit operations. The methodology for defining states and transitions remains the same.

8.1 A family of deterministic Levenshtein-automata for primitive edit operations including merges and splits

We assume that primitive edit operatios are applied in parallel. This means that if V is obtained from $W = x_1 \cdots x_w$ by splitting letter x_i into $x'_i x''_i$, then V must have an occurrence of the split sequence $x'_i x''_i$. We write $d_L^{MS}(V, W)$ for the Levenshtein-distance where merges and splits are treated as primitive edit operations, with $\mathcal{L}_{Lev}^{MS}(n, W)$ we denote the set of all word $V \in \Sigma^*$ where $d_L^{MS}(V, W) \leq n$.

Since primitive edit operations are applied in parallel, the concept of a trace representation can be extended. Splits (resp. merges) are indicated connecting the source letter (letter pair) with its target pair of image letters (target letter).

Remark 8.1.1 Let $W = x_1 x_2 \cdots x_w$ and $V = y_1 y_2 \cdots y_v$ be two words with Levenshtein-distance $n \geq 1$. Assume that neither V is a prefix of W nor vice versa. Let $U = x_1 x_2 \cdots x_i$ (where $0 \leq i \leq v, w$) denote the maximal common prefix of V and W. Then, in any trace representation of a minimal sequence ν of edit operations leading from W to V exactly one of the following five cases holds:

1.-3. *Insertion, substitution, deletion cases.* As in Remark 2.0.3.

4. *Merge case.* The pair $x_{i+1} x_{i+2}$ is connected with y_{i+1}.

5. *Split case.* Letter x_{i+1} is connected with $y_{i+1} y_{i+2}$.

Let $W = x_1 \cdots x_w$ denote the input word and let n denote the degree. As before, the elements of $\{0, 1, \ldots, w\}$ are called *boundaries* of W.

Definition 8.1.2 A *standard position* is an expression of the form $i^{\sharp e}$ where i is a boundary and $0 \leq e \leq n$. An *s-position* is an expression of the form $i_s^{\sharp e}$ where $0 \leq i \leq w-1$ and $1 \leq e \leq n$. A *position* is either a standard position or an s-position. A position π is an *accepting position* iff $\pi = i^{\sharp e}$ is a standard position and if $w - i \leq n - e$.

The intuitive interpretation of expressions $i^{\sharp e}$ is as before. Expressions $i_s^{\sharp e}$ are reached from positions $i^{\sharp e-1}$ under input x'_{i+1} in situations where we have a split $x_{i+1} \mapsto x'_{i+1}x''_{i+1}$.

Definition 8.1.3 Subsumption between positions is explained as follows:

1. A position $i^{\sharp e}$ *subsumes* a position $j^{\sharp f}$ iff $e < f$ and $|j-i| \leq f-e$.
2. A position $i^{\sharp e}$ *subsumes* a position $j_s^{\sharp f}$ iff $f > e$ and $|j-i| \leq f-e$.
3. A position $i_s^{\sharp e}$ *subsumes* a position $j_s^{\sharp f}$ iff $f > e$ and $|j-i| \leq f-e$.

As in the previous cases, with each position π we associate a language $\Phi(\pi)$:

$$\begin{aligned} \Phi(i^{\sharp e}) &:= \mathcal{L}_{Lev}^{MS}(n-e, x_{i+1}\cdots x_w), \\ \Phi(i_s^{\sharp e}) &:= \Sigma \circ \mathcal{L}_{Lev}^{MS}(n-e, x_{i+2}\cdots x_w). \end{aligned}$$

The reader might ask why s-positions do not subsume standard positions. With Definition 8.1.3 we only want to capture subsumption relations that hold in a uniform way. Standard positions π can be accepting positions. In this case, the empty word ε belongs $\Phi(\pi)$. However, our assumption that edit operations are applied in parallel implies that the language associated with an s-position does not contain ε. Hence, there are cases where we do not have a containment on the language side.

Lemma 8.1.4 *Let π and π' denote two distinct positions. If π subsumes π', then $\Phi(\pi') \subseteq \Phi(\pi)$.*

Proof. Assume that standard position $i^{\sharp e}$ subsumes the s-position $j_s^{\sharp f}$. Then $f > e$ and $(f-e)+i \leq j \leq (f-e)+i$. An an example, we consider the case where $i < j$. From $j \leq w-1$ it follows that $i \leq w-2$. Let $V \in \Phi(j_s^{\sharp f})$. Then V has the form zV' where $V' \in \mathcal{L}_{Lev}(n-f, x_{j+2}\cdots x_w)$. The distance between $x_{i+1}\cdots x_w$ and $zx_{i+3}\cdots x_w$ is 1 since merges are primitive edit operations. The distance between $zx_{i+3}\cdots x_w$ and $zx_{j+2}\cdots x_w$ is $j-i-1$. The distance between $zx_{j+2}\cdots x_w$ and zV' is $\leq n-f$. Hence the distance between $x_{i+1}\cdots x_w$ and zV' is $\leq n-f+j-i$. Since $j \leq f-e+i$ the latter distance does not exceed $n-e$ and we have $V \in \Phi(i^{\sharp e})$. The remaining cases are similar. □

Definition 8.1.5 Let $0 \leq i \leq w$. A *state with base position* $i^{\sharp 0}$ is a set M of positions, not necessarily containing $i^{\sharp 0}$, that safisfies the following properties:

1. for each position $j^{\sharp e}$ or $j_s^{\sharp e}$ in M we have $|i-j| \leq e$. I.e., each position of M, with the possible exception of $i^{\sharp 0}$, is subsumed by $i^{\sharp 0}$.
2. M does not contain any position that is subsumed by another element of M.

Definition 8.1.6 Let $W = x_1 \cdots x_w$ as above. Let $\pi := i^{\sharp e}$ be a position, and let $k := \min\{n-e+1, w-i\}$. The *relevant subword of W for position π*, denoted $W_{[\pi]}$, is the subword $x_{i+1} \cdots x_{i+k}$ of W.

Definition 8.1.7 Let W and n as above. An *elementary transition* assigns to each position π and each symbol $x \in \Sigma$ a state $\delta(\pi, x)$. The complete set of elementary transitions is specified in Table 8.1.

(I) $e = 0 < n$	
$i \leq w-2$	$\delta(i^{\sharp e}, x) := \begin{cases} \{(i+1)^{\sharp e}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1, b_2, \ldots, b_k\rangle, \\ \{i^{\sharp e+1}, i_s^{\sharp e+1}, (i+1)^{\sharp e+1}, (i+2)^{\sharp e+1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0, b_2, \ldots, b_k\rangle. \end{cases}$
$i = w-1$	$\delta(i^{\sharp e}, x) := \begin{cases} \{(i+1)^{\sharp e}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1\rangle, \\ \{i^{\sharp e+1}, i_s^{\sharp e+1}, (i+1)^{\sharp e+1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0\rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp e}, x) := \{w^{\sharp e+1}\}$
(II) $0 < e \leq n-1$	
$i \leq w-2$	$\delta(i^{\sharp e}, x) := \begin{cases} \{(i+1)^{\sharp e}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1, b_2, \ldots, b_k\rangle, \\ \{i^{\sharp e+1}, i_s^{\sharp e+1}, (i+1)^{\sharp e+1}, (i+2)^{\sharp e+1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0, b_2, \ldots, b_k\rangle. \end{cases}$ $\delta(i_s^{\sharp e}, x) := (i+1)^{\sharp e}$.
$i = w-1$	$\delta(i^{\sharp e}, x) := \begin{cases} \{(i+1)^{\sharp e}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1\rangle, \\ \{i^{\sharp e+1}, i_s^{\sharp e+1}, (i+1)^{\sharp e+1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0\rangle. \end{cases}$ $\delta(i_s^{\sharp e}, x) := (i+1)^{\sharp e}$.
$i = w$	$\delta(w^{\sharp e}, x) := \{w^{\sharp e+1}\}$
(III) $e = n$	
$i \leq w-1$	$\delta(i^{\sharp n}, x) := \begin{cases} \{(i+1)^{\sharp n}\} \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1\rangle, \\ \emptyset \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0\rangle. \end{cases}$ $\delta(i_s^{\sharp e}, x) := (i+1)^{\sharp e}$.
$i = w$	$\delta(w^{\sharp n}, x) := \emptyset$.

Table 8.1: Table of elementary transitions for $\pi = i^{\sharp e}$.

Lemma 8.1.8 *Let M be a state with base position $i^{\sharp 0}$ where $i < w$. Then the image of any element of M under an elementary transition is always a state with base position $(i+1)^{\sharp 0}$. If M is a state with base position $w^{\sharp 0}$, then the image of any element of M under an elementary transition is always a state with base position $w^{\sharp 0}$.*

Lemma 8.1.9 (Raising Lemma for elementary transitions) *Let $n > 0$ and $1 \leq e \leq n$. Then for any position of degree n of the form $[\pi]^e$ and any $x \in \Sigma$ we have*

$$\delta^{(n)}([\pi]^e, x) = [\delta^{(n-e)}(\pi, x)]^{\sharp e}.$$

In the sequel, with $M \sqcup N$ we denote the reduced union of the states M and N with common base position where we refer to the new notion of subsumption introduced in Definition 8.1.3.

Definition 8.1.10 Let $W = x_1 \cdots x_w$ where $w \geq 0$, let $n \geq 0$. Then $LEV_n^{MS}(W)$ is the deterministic finite state automaton $\langle \Sigma, Q, q_0, F, \Delta \rangle$ where

1. the set of *states* Q contains all states in the sense of Definition 8.1.5,
2. the *initial state* is $q_0 := \{0^{\sharp 0}\}$,
3. the set F of *final states* contains all states $M \in Q$ that contain an accepting position,
4. the *transition function* Δ is defined in the following way: for any symbol $y \in \Sigma$ and any state $M \in Q$, $\Delta(M, y) := \bigsqcup_{\pi \in M} \delta(\pi, y)$.

It follows from Lemma 8.1.8 that Δ assigns to each state $M \in Q$ and each $y \in \Sigma$ again a state $M \in Q$. This shows that $LEV_n^{MS}(W)$ is a deterministic finite state automaton.

The raising lemma holds also in the new situation. We write $\Delta^{(n)}$ for the transition function of $LEV_n^{M-S}(W)$.

Lemma 8.1.11 (Raising Lemma for transitions) *Let $n > 0$ and $1 \leq e \leq n$. Then for state of degree n of the form $[M]^{\sharp e}$ and any $x \in \Sigma$ we have*

$$\Delta^{(n)}([M]^{\sharp e}, x) = [\Delta^{(n-e)}(M, x)]^{\sharp e}.$$

Proof. As for Lemma 4.0.29. □

In the sequel, let $LEV_n^{MS}(W) = \langle \Sigma, Q, q_0, F, \Delta \rangle$ as in Definition 8.1.10.

Proposition 8.1.12 *The following properties hold:*

1. $\mathcal{L}(\emptyset) = \emptyset$,
2. *for all states M, N with a common base position and all $y \in \Sigma$:*
$$\Delta(M \sqcup N, y) = \Delta(M, y) \sqcup \Delta(N, y),$$
3. *for all states M, N with a common base position and all $V \in \Sigma^*$:*
$$\Delta^*(M \sqcup N, V) = \Delta^*(M, V) \sqcup \Delta^*(N, V),$$
4. *for all states $M \subseteq Q \setminus \{\{0^{\sharp 0}\}, \ldots, \{w^{\sharp 0}\}\}$: $\mathcal{L}(M) = \bigcup_{\pi \in M} \mathcal{L}(\{\pi\})$.*

Proof. As for Proposition 4.0.30. □

Proposition 8.1.13 *The following holds:*

$$\begin{array}{rcll} \mathcal{L}(\{i_s^{\sharp e}\}) & = & \Sigma \circ \mathcal{L}_{Lev}^{MS}(n-e, x_{i+2} \cdots x_w) & (0 \leq i \leq w-1, 1 \leq e \leq n) \\ \mathcal{L}(\{i^{\sharp e}\}) & = & \mathcal{L}_{Lev}^{MS}(n-e, x_{i+1} \cdots x_w) & (0 \leq i \leq w, 0 \leq e \leq n) \end{array}$$

Proof. We proceed by induction on n. The case $n = 0$ can be treated as before (cf. proof of Prop. 4.0.31).

Now let $n \geq 1$ and assume that the Proposition is correct for all $0 \leq n' < n$. The case $e = n$ is simple since all relevant transitions are of the form $\{i^{\sharp n}\} \mapsto \{(i+1)^{\sharp n}\}$ under x_{i+1} $(i < w)$ or $\{i_s^{\sharp n}\} \mapsto \{(i+1)^{\sharp n}\}$ under x_{i+1} $(i < w-1)$. Hence assume that $e < n$. Since for $1 \leq e < n$ transitions from states $\{i^{\sharp e}\}$ are defined by raising of transitions of degree $n' = n - e$ the induction hypothesis shows that

$$\begin{aligned} \mathcal{L}(\{i_s^{\sharp e}\}) &= \Sigma \circ \mathcal{L}_{Lev}^{MS}(n-e, x_{i+2}\cdots x_w) \quad (0 \leq i \leq w-1, 2 \leq e \leq n) \quad (8.1) \\ \mathcal{L}(\{i^{\sharp e}\}) &= \mathcal{L}_{Lev}^{MS}(n-e, x_{i+1}\cdots x_w) \quad (0 \leq i \leq w, 1 \leq e \leq n) \quad (8.2) \end{aligned}$$

It remains to prove that

$$\begin{aligned} \mathcal{L}(\{i_s^{\sharp 1}\}) &= \Sigma \circ \mathcal{L}_{Lev}^{MS}(n-1, x_{i+2}\cdots x_w) \quad (0 \leq i \leq w-1) \quad (8.3) \\ \mathcal{L}(\{i^{\sharp 0}\}) &= \mathcal{L}_{Lev}^{MS}(n, x_{i+1}\cdots x_w) \quad (0 \leq i \leq w) \quad (8.4) \end{aligned}$$

In the sequel, let W_i denote the suffix $x_{i+1}\cdots x_w$ of W $(0 \leq i \leq w)$. From (8.2) we obtain the following : for all positions $i^{\sharp e}$ and $j^{\sharp f}$ such that $e \neq 0 \neq f$: if $i^{\sharp e}$ is subsumed by $j^{\sharp f}$, then $\mathcal{L}(\{i^{\sharp e}\})$ is a proper subset of $\mathcal{L}(\{j^{\sharp f}\})$ (††).

I. We first show that $\Sigma \circ \mathcal{L}_{Lev}^{MS}(n-1, x_{i+2}\cdots x_w) \subseteq \mathcal{L}(\{i_s^{\sharp 1}\})$. The elementary transitions show that from $\{i_s^{\sharp 1}\}$ we reach $\{(i+1)^{\sharp 1}\}$ under any $x \in \Sigma$. (8.2) shows that $\Sigma \circ \mathcal{L}_{Lev}^{MS}(n-1, x_{i+2}\cdots x_w) \subseteq \mathcal{L}(\{i_s^{\sharp 1}\})$.

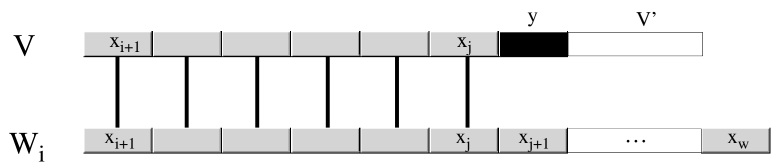
II. We show that $\mathcal{L}(\{i_s^{\sharp 1}\}) \subseteq \Sigma \circ \mathcal{L}_{Lev}^{MS}(n-1, W_{i+2})$. With any input $x \in \Sigma$, from $\mathcal{L}(\{i_s^{\sharp 1}\})$ we reach $\{(i+1)^{\sharp 1}\}$. By (8.2), $\mathcal{L}(\{(i+1)^{\sharp e}\}) = \mathcal{L}_{Lev}^{MS}(n-1, x_{i+2}\cdots x_w)$. It follows that $\mathcal{L}(\{i_s^{\sharp 1}\}) \subseteq \Sigma \circ \mathcal{L}_{Lev}^{MS}(n-1, W_{i+2})$.

III. We show that $\mathcal{L}_{Lev}^{MS}(n, W_i) \subseteq \mathcal{L}(\{i^{\sharp 0}\})$. Let $V \in \mathcal{L}_{Lev}^{MS}(n, W_i)$.

Case 1.1: V is obtained from W_i by deleting a suffix of length k $(0 \leq k \leq n)$ of W_i. Starting from state $\{i^{\sharp 0}\}$ and consuming V we reach state $\{(w-k)^{\sharp 0}\}$. Since $(w-k)^{\sharp 0}$ is an accepting position, state $\{(w-k)^{\sharp 0}\}$ is final, hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

Case 1.2: W_i is obtained from V by deleting a suffix of length k $(1 \leq k \leq n)$ of V. Starting from state $\{i^{\sharp 0}\}$ and first consuming W_i we reach $\{w^{\sharp 0}\}$. The k additional transitions lead to $\{w^{\sharp k}\}$. Since $w^{\sharp k}$ is an accepting position, the latter state is final, hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

Case 2: In the remaining cases there exists an index $j < w$ such that $V = x_{i+1}\cdots x_j y V'$ where $y \neq x_{j+1}$.



We have $yV' \in \mathcal{L}_{Lev}^{MS}(n, x_{j+1}\cdots x_w)$. We fix a sequence ν of edit operations leading from $x_{j+1}\cdots x_w$ to yV' of minimal length and consider the five cases described in Remark 8.1.1.

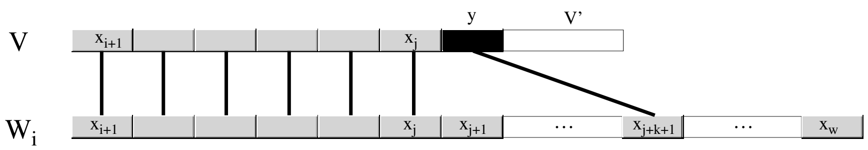
2.1. If the occurrence of y is an insertion before x_{j+1} (where $i \leq j \leq w$), then $V' \in \mathcal{L}_{Lev}(n-1, x_{j+1}\cdots x_w)$. Since $y \neq x_{j+1}$, starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots, \{j^{\sharp 0}\}, M$ where M contains $j^{\sharp 1}$ (cf. elementary transitions). It follows from (8.2) and Part 4 of Proposition 8.1.12 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.2. If the occurrence of y substitutes x_{j+1} (where $i \leq j < w$), then V' belongs to $\mathcal{L}_{Lev}(n-1, x_{j+2} \cdots x_w)$. Starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots, \{j^{\sharp 0}\}, M$ where M contains $(j+1)^{\sharp 1}$. It follows from (8.2) and Part 4 of Proposition 8.1.12 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.3. If the occurrence of y is caused by a merge of x_{j+1} and x_{j+2} (where $j \leq w-2$), then $V' \in \mathcal{L}^{MS}_{Lev}(n-1, x_{j+3} \cdots x_w)$. Since $y \neq x_{j+1}$, starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots, \{j^{\sharp 0}\}, M$ where M contains $(j+2)^{\sharp 1}$ (cf. elementary transitions). The induction hypothesis (8.2) and Part 4 of Proposition 8.1.12 show that $V' \in \mathcal{L}(M)$. It follows that $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.4. If the occurrence of y is caused by a split $x_{i+1} \mapsto yy'$, then V' has the form $y'V''$ where $V'' \in \mathcal{L}^{MS}_{Lev}(n-1, x_{j+2} \cdots x_w)$. Starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we arrive at state M containing $j_s^{\sharp 1}$. It follows from Parts I and II that $\mathcal{L}(\{j_s^{\sharp 1}\}) = \Sigma \circ \mathcal{L}^{MS}_{Lev}(n-1, x_{j+2} \cdots x_w)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.5. In the remaining case there exists some $1 \leq k \leq n$ such that we have a stroke from x_{j+k+1} to y in the trace representation of ν.



We have $j \leq w-2$ and the k letters $x_{j+1}, x_{j+2}, \ldots, x_{j+k}$ are erased. The distance between $x_{j+k+1} \cdots x_w$ and yV' is bounded by $n-k$. It follows from the definition of elementary transitions (cf. Table 8.1) that we reach state $M := \{j^{\sharp 1}, j_s^{\sharp 1}, (j+1)^{\sharp 1}, (j+2)^{\sharp 1}\}$ from $\{j^{\sharp 0}\}$ by consuming x_{j+k+1}.

2.5.1. Assume first that $x_{j+k+1} = y$. Then the distance between $x_{j+k+2} \cdots x_w$ and V' is bounded by $n-k$. By (8.2), $V' \in \mathcal{L}(\{(j+k+1)^{\sharp k}\})$. Since $(j+k+1)^{\sharp k}$ is subsumed by $(j+2)^{\sharp 1}$ also $V' \in \mathcal{L}(\{(j+2)^{\sharp 1}\})$, by (††), hence $V' \in \mathcal{L}(M)$ by Part 4 of Proposition 8.1.12. It follows that $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

2.5.2. Assume that $x_{j+k+1} \neq y$. Then the distance between $x_{j+k+2} \cdots x_w$ and V' is bounded by $n-k-1$. Starting from state $\{i^{\sharp 0}\}$ and consuming the letters $x_{i+1}, \ldots x_j, y$ we eventually reach a state M containing $(j+1)^{\sharp 1}$. This position subsumes $(j+k+1)^{\sharp k+1}$. It follows from (8.2), (††) and Part 4 of Proposition 8.1.12 that $V' \in \mathcal{L}(M)$. Hence $V \in \mathcal{L}(\{i^{\sharp 0}\})$.

IV. It remains to prove that $\mathcal{L}(\{i^{\sharp 0}\}) \subseteq \mathcal{L}^{MS}_{Lev}(n, W_i)$ for $0 \leq i \leq w$. Let $V \in \mathcal{L}(\{i^{\sharp 0}\})$. If V is accepted - starting from $\{i^{\sharp 0}\}$ - on a path of singleton sets with basic positions $\{(i+1)^{\sharp 0}\}, \{(i+2)^{\sharp 0}\}, \ldots, \{k^{\sharp 0}\}$, then $k^{\sharp 0}$ is accepting, which implies that V has the form $x_{i+1} \cdots x_k$ where $w-k \leq n$. This shows that $V \in \mathcal{L}_{Lev}(n, W_i)$. In the other case, starting from state $\{i^{\sharp 0}\}$ and consuming the prefix $V'y$ of $V = V'yV''$ we reach states $\{(i+1)^{\sharp 0}\}, \ldots \{j^{\sharp 0}\}, M$ where $M \neq \{(j+1)^{\sharp 0}\}$.

Case (a): $j < w-2$ and M has the form $\{j^{\sharp 1}, j_s^{\sharp 1}, (j+1)^{\sharp 1}, (j+2)^{\sharp 1}\}$. The induction hypothesis (8.1) and (8.2) and Part IV of Lemma 8.1.12 show that V''

belongs to

$$\begin{array}{ll} & \mathcal{L}_{Lev}^{MS}(n-1, W_j) \\ \cup & \Sigma \circ \mathcal{L}_{Lev}^{MS}(n-1, W_{j+1}) \\ \cup & \mathcal{L}_{Lev}^{MS}(n-1, W_{j+1}) \\ \cup & \mathcal{L}_{Lev}^{MS}(n-1, W_{j+2}). \end{array}$$

Treating y as in insertion (resp. the first letter of a split, a substitution, a merged symbol) it easily follows that $V \in \mathcal{L}_{Lev}^{MS}(n, W_i)$.

Case (b): $j < w-1$ and M has the form $\{j^{\sharp 1}, j_s^{\sharp 1}, (j+1)^{\sharp 1}\}$. Similar to case (a).

Case (c): $j = w$. In this case $M = \{w^{\sharp 1}\}$ and $V' = W_i$. It follows from (8.2) that V'' has distance $\leq n-1$ to the empty word ε. Hence V has distance $\leq n$ to W_i. □

From Proposition 7.1.13 we obtain the parallel result to Theorem 4.0.32.

Theorem 8.1.14 *$LEV_n^{MS}(W)$ is a deterministic and acyclic Levenshtein-automaton of degree n for W for primitive edit operations including merges and splits. For fixed degree n, the size of $LEV_n^{MS}(W)$ is linear in $|W|$.*

8.2 Computation of deterministic Levenshtein-automata for primitive edit operations including merges and splits

As in the previous cases, the general description of the Levensthein-automaton $LEV_n^{MS}(W)$ can be used to derive, for any fixed number n, an algorithm that actually computes the automaton $LEV_n^{MS}(W)$ in linear time, given any input word W. The principle will be illustrated for degree $n = 1$.

For input $W = x_1 \cdots x_w$ and $n = 1$ the list of positions is

$$\begin{array}{l} 0^{\sharp 0}, \ldots, w^{\sharp 0}, 0^{\sharp 1}, \ldots, w^{\sharp 1} \\ 0_t^{\sharp 1}, \ldots, (w-2)_t^{\sharp 1}. \end{array}$$

It suffices to consider the following states:

$$\begin{array}{rcl} \emptyset & & \text{failure state}, \\ A_i & := & \{i^{\sharp 0}\} \quad (0 \leq i \leq w), \\ B_i & := & \{i^{\sharp 1}\} \quad (0 \leq i \leq w), \\ C_i & := & \{i^{\sharp 1}, (i+1)^{\sharp 1}\} \quad (0 \leq i \leq w-1), \\ D_i & := & \{i^{\sharp 1}, (i+2)^{\sharp 1}\} \quad (0 \leq i \leq w-2), \\ E_i & := & \{i^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \quad (0 \leq i \leq w-2), \\ F_i & := & \{i^{\sharp 1}, i_s^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \quad (0 \leq i \leq w-2), \\ G_i & := & \{i^{\sharp 1}, i_s^{\sharp 1}, (i+1)^{\sharp 1}\} \quad (0 \leq i \leq w-1), \end{array}$$

The initial state is A_0. Accepting positions are $w^{\sharp 1}$, $w^{\sharp 0}$, as well as $(w-1)^{\sharp 0}$ for $w \geq 1$. It follows immediately that the final states are

$$A_w, A_{w-1}, B_w, C_{w-1}, D_{w-2}, E_{w-2}, F_{w-2}, G_{w-1} \qquad \text{for } w \geq 2,$$

$$\begin{aligned} A_w, A_{w-1}, B_w, C_{w-1}, G_{w-1} \qquad & \text{for } w = 1, \\ A_w, B_w \qquad & \text{for } w = 0. \end{aligned}$$

For the case $n = 1$, we obtain the set of elementary transitions given in Table 8.2. Using Table 8.2 it is simple to compute a parametric description of the full transition

(I) $e = 0$	
$i \leq w-2$	$\delta(i^{\sharp 0}, x) := \begin{cases} \{(i+1)^{\sharp 0}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1, b_2 \rangle, \\ \{i^{\sharp 1}, i_s^{\sharp 1}, (i+1)^{\sharp 1}, (i+2)^{\sharp 1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0, b_2 \rangle. \end{cases}$
$i = w-1$	$\delta(i^{\sharp 0}, x) := \begin{cases} \{(i+1)^{\sharp 0}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1 \rangle, \\ \{i^{\sharp 1}, i_s^{\sharp 1}, (i+1)^{\sharp 1}\} \\ \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0 \rangle. \end{cases}$
$i = w$	$\delta(w^{\sharp 0}, x) := \{w^{\sharp 1}\}$
(III) $e = 1$	
$i \leq w-1$	$\delta(i^{\sharp 1}, x) := \begin{cases} \{(i+1)^{\sharp 1}\} \quad \text{for } \chi(x, W_{[\pi]}) = \langle 1 \rangle, \\ \emptyset \quad \text{for } \chi(x, W_{[\pi]}) = \langle 0 \rangle. \end{cases}$ $\delta(i_s^{\sharp 1}, x) := (i+1)^{\sharp 1}$.
$i = w$	$\delta(w^{\sharp 1}, x) := \emptyset$.

Table 8.2: Table of elementary transitions for $n = 1$, primitive edit operations including merges and splits.

function Δ as before. The description is given in Table 8.3.

Obviously, given the above generic description of LEV_1^{MS} it is possible to generate for any concrete input W the automaton $LEV_1^{MS}(W)$ in time $O(|W|)$.

Theorem 8.2.1 *There exists an algorithm that computes for any input word W the automaton $LEV_1^{MS}(W)$ in time and space $O(|W|)$.*

Corollary 8.2.2 *For any input W, the minimal deterministic Levenshtein-automaton of degree 1 for W where primitive edit operations include merges and splits can be computed in time and space $O(|W|)$.*

Example 8.2.3 Figure 8.1 describes the automaton $LEV_1^{MS}(W)$ for the input word "atlas". For each word W of length 5 with 3-profile sequence $(1,2,3), (1,2,3), (1,2,3)$ the automaton $LEV_1^{MS}(W)$ has the same structure, modulo transition labels. Similarly Figure 8.2 describes the structure of $LEV_1^{MS}(W)$ for the word "otter". Here for each word W of length 5 with 3-profile sequence $(1,2,2), (1,1,2), (1,2,3)$ the automaton $LEV_1^{MS}(W)$ has the same structure, modulo transition labels.

Theorem 8.2.4 *For any fixed degree n, there exists an algorithm that computes for any input word W the automaton $LEV_n^{MS}(W)$ in time and space $O(|W|)$.*

Corollary 8.2.5 *For any input W, the minimal deterministic Levenshtein-automaton of degree n for W where primitive edit operations include merges and splits can be computed in time and space $O(|W|)$.*

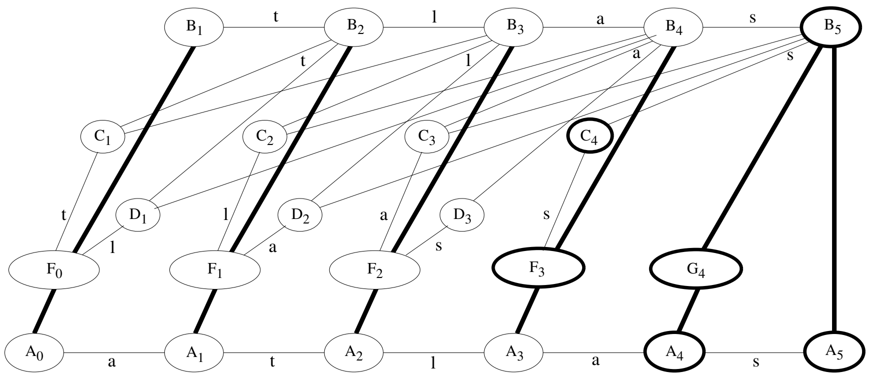


Figure 8.1: Deterministic Levenshtein-automaton $LEV_1^{MS}(W)$ for input W = “atlas”.

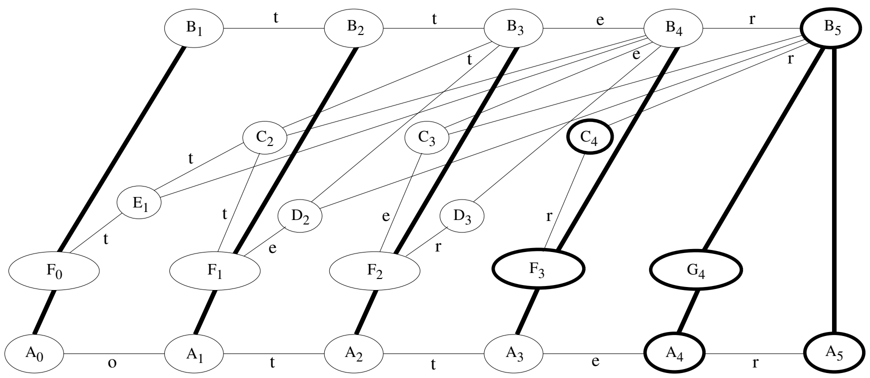


Figure 8.2: Deterministic Levenshtein-automaton $LEV_1^{MS}(W)$ for input W = “otter”.

$0 \leq i \leq w-3$							
$\chi(x, x_{i+1}x_{i+2}x_{i+3})$	A_i	B_i	C_i	D_i	E_i	F_i	G_i
$\langle 0,0,0\rangle$	F_i	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	B_{i+1}	B_{i+1}
$\langle 1,0,0\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}
$\langle 0,1,0\rangle$	F_i	$\emptyset$	B_{i+2}	$\emptyset$	B_{i+2}	C_{i+1}	C_{i+1}
$\langle 0,0,1\rangle$	F_i	$\emptyset$	$\emptyset$	B_{i+3}	B_{i+3}	D_{i+1}	B_{i+1}
$\langle 1,1,0\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	B_{i+1}	C_{i+1}	C_{i+1}	C_{i+1}
$\langle 1,0,1\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	D_{i+1}	D_{i+1}	D_{i+1}	B_{i+1}
$\langle 0,1,1\rangle$	F_i	$\emptyset$	B_{i+2}	B_{i+3}	C_{i+2}	E_{i+1}	C_{i+1}
$\langle 1,1,1\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	D_{i+1}	E_{i+1}	E_{i+1}	C_{i+1}

$i = w-2$							
$\chi(x, x_{i+1}x_{i+2})$	A_i	B_i	C_i	D_i	E_i	F_i	G_i
$\langle 0,0\rangle$	F_i	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	B_{i+1}	B_{i+1}
$\langle 1,0\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}
$\langle 0,1\rangle$	F_i	$\emptyset$	B_{i+2}	$\emptyset$	B_{i+2}	C_{i+1}	C_{i+1}
$\langle 1,1\rangle$	A_{i+1}	B_{i+1}	C_{i+1}	B_{i+1}	C_{i+1}	C_{i+1}	C_{i+1}

$i = w-1$				
$\chi(x, x_{i+1})$	A_i	B_i	C_i	G_i
$\langle 0\rangle$	G_i	$\emptyset$	$\emptyset$	B_{i+1}
$\langle 1\rangle$	A_{i+1}	B_{i+1}	B_{i+1}	B_{i+1}

$i = w$		
$\chi(x, \varepsilon)$	A_i	B_i
$\langle\rangle$	B_i	$\emptyset$

Table 8.3: Transitions of $LEV_1^{MS}(W)$.

8.3 Experimental results

Experimental results were made using a Bulgarian lexicon BL with 870,000 word entries and a dictionary of german composite nouns GL with 6,058,198 entries. The following algorithms were implemented in C and tested on a 500 MHz (BL) resp. 600 MHz (GL) Pentium III machine under Linux:

- The algorithm for computing, given input W, the automaton $LEV_n(W)$ ($n = 1,2,3$),
- the correction algorithm based on Levenshtein-automata described in Section 3 ($n = 1,2,3$),
- the correction algorithm based on imitation of Levensthein-automata described in Section 6 ($n = 1,2,3$),
- the variants of the above algorithms for the modified Levenshtein distances where transpositions (resp. merges and splits) are treated as additional edit operations (see [?] for a detailled description of Levenshtein-automata for these modified distances).

Evaluation of correction with BL

For the Bulgarian lexicon BL we used the prefixes of length $3, 4, \ldots, 19$ of all dictionary words as garbled input "words" and computed the correction candidates. The number of test words of each length can be seen from the following table.

Length	3	4	5	6	7	8
♯ prefixes	3,152	12,121	30,243	59,835	101.763	150,046
Length	9	10	11	12	13	14
♯ prefixes	190,318	203,520	184,138	139,982	91,252	52,603
Length	15	16	17	18	19	20
♯ prefixes	27,997	14,763	8.179	4.601	2.790	1.585

The tables given below describe the results for

1. correction with BL and standard Levenshtein-distance with bound $n = 1, 2, 3$ (Tables 8.4, 8.5 and 8.6),
2. correction with BL and Levenshtein-distance where transpositions are treated as primitive edit operations, with bounds $n = 1, 2, 3$ (Table 8.7),
3. correction with BL and Levenshtein-distance where merges and splits are treated as primitive edit operations, with bound $n = 1, 2, 3$ (Table 8.8).

In Tables 8.4, 8.5 and 8.6, column 1 gives the length of the input words. Column 2 (LA) describes the average time that is needed to compute the Levenshtein-automaton for an input word. Column 3 describes the average time that is needed for parallel traversal (PT) of dictionary automaton and Levenshtein-automaton. Column 4 (TCT1) gives the average total correction time for the correction method based on computation of Levenshtein-automata. Column 6 (TCT2) gives the average total correction time for the correction method based on imitation of Levenshtein-automata. Column 7 (NC) yields the average number of correction candidates per word. Times are in milliseconds. It is important to note that the time that is needed to output the correction candidates is always included.

As a résumé, the second correction method based on simulation of Levenshtein-automata is more efficient. The smaller number of correction candidates for large prefixes leads to the effect that for prefixes of length > 13 correction times decrease for longer input words when using the second correction method. The use of transpositions as primitive edit operations does not influence correction times and the number of correction candidates in a significant way. In contrast, much more search is needed when treating merges and splits as additional primitive edit operations. Both correction times and number of correction candidates grow.

Evaluation of correction with GL

Table 8.9 describes the results for correction with the german dictionary of composite nouns GL with $6,058,198$ entries. For each length $l = 5, \ldots, 19$, we randomly selected $1,000$ prefixes of length l of entries and computed for each prefix all entries of GL where the standard Levenshtein-distance does not exceed bound $n = 1, 2, 3$. We give the correction time (including output of correction candidates) and the average number of corrections.

Length	LA	PT	TCT1	TCT2	NC
3	0.228	0.152	0.381	0.324	10.31
4	0.251	0.183	0.434	0.351	8.66
5	0.272	0.201	0.473	0.351	6.97
6	0.294	0.213	0.507	0.355	6.66
7	0.318	0.222	0.540	0.362	6.44
8	0.340	0.233	0.573	0.375	6.34
9	0.362	0.244	0.607	0.389	5.84
10	0.385	0.255	0.639	0.403	5.25
11	0.410	0.261	0.671	0.413	4.65
12	0.430	0.270	0.700	0.421	4.01
13	0.452	0.273	0.726	0.424	3.61
14	0.474	0.273	0.748	0.422	3.24
15	0.497	0.270	0.767	0.414	2.98
16	0.520	0.266	0.786	0.404	2.73
17	0.544	0.260	0.804	0.400	2.62
18	0.565	0.263	0.828	0.398	2.51
19	0.588	0.262	0.849	0.394	2.35

Table 8.4: Results for BL, standard Levenshtein-distance, bound $n = 1$.

Length	LA	PT	TCT1	TCT2	NC
3	1.40	2.18	3.57	3.19	227
4	1.62	2.56	4.18	3.68	175
5	1.78	2.76	4.55	3.74	111
6	1.94	2.85	4.80	3.75	76.2
7	2.10	2.91	5.02	3.65	57.3
8	2.26	3.00	5.26	3.64	48.2
9	2.42	3.08	5.50	3.66	39.1
10	2.58	3.16	5.74	3.72	32.2
11	2.74	3.22	5.95	3.77	26.2
12	2.89	3.25	6.15	3.80	20.7
13	3.06	3.27	6.33	3.81	16.3
14	3.21	3.25	6.46	3.77	12.8
15	3.37	3.19	6.56	3.70	10.7
16	3.53	3.12	6.65	3.63	9.19
17	3.68	3.08	6.77	3.58	8.29
18	3.84	3.05	6.89	3.54	7.84
19	4.00	3.03	7.03	3.52	7.35

Table 8.5: Results for BL, standard Levenshtein-distance, bound $n = 2$.

Length	LA	PT	TCT1	TCT2	NC
3	13.3	11.6	25.0	16.1	2411
4	14.8	13.8	28.6	18.9	2108
5	16.3	15.0	31.4	20.4	1397
6	18.0	15.7	33.7	21.2	852
7	19.2	16.0	35.2	20.9	538
8	20.3	16.4	36.7	20.9	381
9	21.5	16.7	38.2	20.6	269
10	22.8	16.9	39.7	20.4	192
11	23.9	17.1	41.1	20.3	138
12	27.6	22.8	50.4	25.6	96.0
13	30.9	21.7	52.6	25.2	63.7
14	32.2	21.9	54.2	25.1	41.8
15	33.6	20.9	54.5	24.0	28.6
16	34.8	21.2	56.0	24.0	21.8
17	36.8	20.1	56.9	23.8	18.3
18	38.1	19.8	57.8	23.0	16.0
19	39.0	19.9	58.9	22.9	14.6

Table 8.6: Results for BL, standard Levenshtein-distance, bound $n = 3$.

Length	$n = 1$ time	NC	n=2 time	NC	$n = 3$ time	NC
3	0.330	10.4	4.03	231	17.0	2143
4	0.362	8.75	4.58	178	21.0	2139
5	0.357	7,04	4.67	114	23.5	1428
6	0.360	6.69	4.69	77.3	21.5	869
7	0.367	6.46	4.57	57.8	21.2	547
8	0.379	6.36	4.58	48.6	21.5	386
9	0.394	5.85	4.62	39.4	21.1	272
10	0.407	5.25	4.69	32.3	22.6	194
11	0.416	4.66	4.85	26.4	20.9	140
12	0.428	4.10	4.78	20.8	20.7	96.8
13	0.430	3.62	4.78	16.3	20.5	64.1
14	0.428	3.24	4.73	12.8	21.9	42.0
15	0.424	2.98	4.74	10.8	21.3	26.7
16	0.416	2.73	4.59	9.21	20.9	21.8
17	0.410	2.62	4.54	8.30	21.0	18.3
18	0.404	2.51	4.50	7.85	24.5	15.9
19	0.405	2.35	4.43	7.36	26.7	14.6

Table 8.7: Results for BL, Levenshtein-distance where transpositions are treated as primitive edit operations, bounds $n = 1, 2, 3$, times in milliseconds.

Length	$n = 1$ time	NC	$n = 2$ time	NC	$n = 3$ time	NC
3	1.86	48.1	30.3	3216	139	$> 10^4$
4	1.79	31.6	32.3	2125	161	$> 10^4$
5	1.79	21.3	33.7	1195	167	9998
6	1.78	16.7	34.0	667	175	8050
7	1.82	14.8	31.8	404	182	8106
8	2.32	13.9	31.4	278	186	5606
9	2.38	12.6	31.5	197	184	3654
10	2.41	11.1	31.8	144	174	2291
11	2.43	9.64	34.2	107	168	1433
12	2.47	8.19	34.3	77.5	169	872
13	2.47	6.88	33.8	53.8	175	493
14	2.45	5.85	35.4	36.8	171	257
15	2.39	5.20	30.6	26.1	170	116
16	2.34	4.65	30.1	20.2	165	54.7
17	2.29	4.26	29.7	17.0	166	35.8
18	2.25	4.17	29.5	14.8	166	27.9
19	2.22	3.85	29.3	13.7	163	24.3

Table 8.8: Results for BL, Levenshtein-distance where merges and splits are treated as primitive edit operations, bounds $n = 1, 2, 3$, times in milliseconds.

Length	$n = 1$ time	NC	$n = 2$ time	NC	$n = 3$ time	NC
5	1.33	4.41	41.5	51.3	313	434
6	1.44	3.40	42.5	34.7	321	337
7	1.59	2.95	39.9	21.9	307	216
8	1.63	2.71	38.7	13.1	307	120
9	1.66	2.48	40.4	9.86	306	80.2
10	1.73	2.32	39.0	7.46	288	50.3
11	1.77	2.14	39.5	6.15	290	36.7
12	1.82	2.01	39.1	4.67	288	23.8
13	1.85	1.89	39.8	4.34	293	19.5
14	1.89	1.80	40.2	3.72	296	14.2
15	1.92	1.71	40.2	3.17	295	10.7
16	1.95	1.65	40.0	2.82	291	7.77
17	1.99	1.60	38.9	2.52	285	6.22
18	2.02	1.56	38.3	2.37	281	5.36
19	2.04	1.34	37.8	1.89	274	3.77

Table 8.9: Results for GL, standard Levenshtein-distance, bounds $n = 1, 2, 3$, times in milliseconds.

Remark 8.3.1 Oflazer gives the following average correction times for a german dictionary with $174,573$ words. For distance bound $n = 1$, 27.09 milliseconds, for $n = 2$, 169.88 milliseconds, for $n = 3$, 582.45 milliseconds. Oflazer's experiments were made on a SPARCstation 10/41. Since we used for our test series a faster machine on the one-hand side, but much larger dictionaries on the other side, an exact comparison of both approaches is impossible. We think, however, that our results show that our method is clearly superior in terms of efficiency.

Chapter 9

Conclusion

We introduced two related methods for correcting garbled words using an electronic dictionary that is implemented as a deterministic finite state automaton. The correction procedures are similar to Oflazer's approach [Ofl96], but completely avoid the computation of Levenshtein-distances. Instead, Levenshtein-automata for the input words are used to control lexical search. We have shown that appropriate deterministic Levenshtein-automata can be computed in time linear in the length of the input. Our second method shows that even the actual computation of a deterministic Levenshtein-automaton for the input word can be avoided since precompiled tables may be used to simulate transitions in the automaton. The experimental results show that our techniques lead to a very fast selection of correction candidates for garbled words.

The complexity results for computing the (minimal) deterministic Levenstheinautomaton for a given input word immediately lead to the following side results.

Lemma 9.0.2 *For any fixed number n, given two words W and V of length w and v respectively, it is decidable in time $O(\max(w, v))$ if the Levenshtein-distance between W and V is $\leq n$.*

Lemma 9.0.3 *For any fixed number n, given a text of words of length h and a word W of length w we can compute in time $O(\max(h, w))$ all words V of the text where the Levenshtein-distance between V and W does not exceed n.*

The results obtained in this paper should be extended in several directions. The situation should be considered where edit operations come with specific costs that depend on the symbols of the operation. Eventually, in application scenarios different methods for ranking correction candidates should be tested that take the frequency of occurrences of a given correction candidate into account.

As to related work, two other approaches, both using methods from automata theory for string correction should be mentioned. Bunke [Bun93] has shown that for any given word W the columns of the table computed in the Fisher-Wagner algorithm can be compiled into a deterministic finite state automaton. For any word V the automaton may be used to compute the Levenshtein-distance between V and W in time linear in the length $|V|$ of V. Given a dictionary of words $W_1, \ldots, W_d$, a similar automaton can be given that computes the Levenshtein-distance between V

and each of the words W_i in time $O(|V|)$. The problem with the approach is that the size of the automaton is exponential in the sum of the length of the words in the dictionary. Hence the approach can only be used for very small dictionaries.

Another interesting approach is described in [CSY99]. Recall that we assign to each input word a Levenshtein-automaton and leave the dictionary automaton unmodified. In [CSY99] a construction is given for computing, given a finite state automaton A, a lifted version A^n that accepts all words V that have Levenshtein-distance $\leq n$ to some word accepted by A.[1] In principle, the construction can be used to lift a dictionary automaton A in order to compute a ,,correction transducer" A^n that yields, given input V, all dictionary words with Levenshtein-distance $\leq n$ to V. Assuming that A_n is deterministic, a run — hence correction of an input word — does not involve any search, or backtracking. However, determinization of a non-deterministic correction transducer is likely be too space-comsuming for large dictionaries. Nevertheless, is seems promising to consider variants of the techniques described in [CSY99] for lexical correction.

[1]This description is simplified. In [CSY99] distinct metrics for defining neighbourhoods are considered, and a generalization of finite state automata is used, so-called ,,lexical analyzers".

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